PREFACE

In a bid to standardise higher education in the country, the University Grants Commission (UGC) has introduced Choice Based Credit System (CBCS) based on five types of courses: core, generic discipline specific elective, and ability/ skill enhancement for graduate students of all programmes at Elective/ Honours level. This brings in the semester pattern, which finds efficacy in tandem with credit system, credit transfer, comprehensive and continuous assessments and a graded pattern of evaluation. The objective is to offer learners ample flexibility to choose from a wide gamut of courses, as also to provide them lateral mobility between various educational institutions in the country where they can carry acquired credits. I am happy to note that the University has been recently accredited by National Assessment and Accreditation Council of India (NAAC) with grade "A".

UGC (Open and Distance Learning programmes and Online Programmes) Regulations, 2020 have mandated compliance with CBCS for all the HEIs in this mode. Welcoming this paradigm shift in higher education, Netaji Subhas Open University (NSOU) has resolved to adopt CBCS from the academic session 2021-22 at the Under Graduate Degree Programme level. The present syllabus, framed in the spirit of syllabi recommended by UGC, lays due stress on all aspects envisaged in the curricular framework of the apex body on higher education. It will be imparted to learners over the six semesters of the Programme.

Self Learning Materials (SLMs) are the mainstay of Student Support Services (SSS) of an Open University. From a logistic point of view, NSOU has embarked upon CBCS presently with SLMs in English. Eventually, these will be translated into Bengali too, for the benefit of learners. As always, we have requisitioned the services of the best academics in each domain for the preparation of new SLMs, and I am sure they will be of commendable academic support. We look forward to proactive feedback from all stake-holders who will participate in the teaching-learning of these study materials. It has been a very challenging task well executed, and 1 congratulate all concerned in the preparation of these SLMs.

I wish the venture a grand success.

Professor (Dr.) Subha Sankar Sarkar

Vice-Chancellor

Netaji Subhas Open University
Choice Based Credit System (CBCS)
Honours in Mathematics (HMT)
Course : Dianamical System
Course Code : GE-MT-21

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Netaji Subhas
Open University

UG : Mathematics
(HMT)

Course : Dianamical System
Course Code : GE-MT-21

Unit 1 ❑ First Order Equations

Structure

1.0 Objective

In a classical treatment of ordinary differential equations, the main attention is to find the solution curve. The purpose of this unit is to develop some elementary yet important examples of first-order differential equations. These examples illustrate some of the basic ideas in the theory of ordinary differential equations in the simplest possible settings. Some examples of existence and uniqueness of solution are included to provide the basic idea when solution of differential equation exists uniquely. The basic models such as the logistic models with harvesting, are included to give the reader a taste of certain topics, e.g. bifurcations, periodic solutions that we will return often in other units.

1.1 Introduction

A first order differential equation is of the form $F\left(t, x, \frac{dx}{dt}\right) = 0$, where t is independent variable, x is the dependent variable and F is a given function. Sometimes the equation can be written in the normal form as $\frac{dx}{dt} = f(t, x)$

A solution of the equation is of the form $x = x(t,c)$, where c is an arbitrary constant. This solution is called general solution. There are several well known methods namely separation of variables, variation of parameters, methods using integrating factors etc. for solving first order ODE. There are many techniques to solve linear ODEs but there is no general method for solving even a first order nonlinear ODE.

Some typical nonlinear first order ODEs can be solved by Bernoulli's method (1697), method of separation of variables, method of variation of parameters, method using integrating factors etc. The lack of general formula for solving nonlinear ODEs has two important consequences. Firstly, methods which yield approximate solution (numerical) and give qualitative information about solutions assume greater significance for nonlinear equations. Secondly, questions dealing with the existence and uniqueness of solutions became important. The following questions arias naturally :

- Given an IVP is there a solution to it (question of existence)?
- If there is a solution, is the solution unique (question of uniqueness)?
- For which values of x does the solution to IVP exists (the interval of existence)?

1.2 Existence and Uniqueness of Solution

Consider the initial value problem (IVP) as

$$\frac{dx}{dt} = f(t,x) \text{ with the initial condition } x(t_0) = x_0. \qquad (1.1)$$

THEOREM 1.2.1 (Existence theorem): Suppose that f(t,x) is continuous function in some region $R = \{(t, x) : |t - t_0| < a, |x - x_0| < b\}$ $(a, b > 0)$. If f is continuous in $R \forall (t,x) \in R$, then there exists at least one solution $x = x(t)$ of (1.1) in the interval $|t - t_0| \leq \alpha$ where $\alpha = \min \{a,b/k\}$. Notice that here R is a subset of real plane.

Theorem 1.2.2 (Uniqueness theorem): Suppose f and $\left|\frac{\partial f}{\partial x}\right|$ are continuous functions in R. Hence both f and $\left|\frac{\partial f}{\partial x}\right|$ are bounded in R, i.e $|f(t,x)| \leq k$ and $\left|\frac{\partial f}{\partial x}\right| \leq C, \forall (t,x) \in R$. Then the IVP (1.1) has at-most one solution $x = x(t)$ defined in the interval $|t - t_0| \leq \alpha$ where $\alpha = \min\{a, b/k\}$.

Combining Theorem 1.2.1 and Theorem 1.2.2, the IVP (1.1) has unique solution x = x(t) in $|x - x_0| \leq \alpha$, $|x - x_0| \leq b$.

1.3. Lipschitz Condition

Now, the condition $\left|\frac{df}{dx}\right| \leq C$ of the above theorem can be replaced by a weaker condition which is known as Lipschitz condition.

A function f(t,x) is said to satisfy Lipschitz condition in R if there exists a constant C such that $|f(t,x_1) - f(t,x_2)| \leq C|x_1 - x_2|$, $\forall (t,x_1), (t,x_2) \in R$.

Note l: The existence and uniqueness theorem stated above are local in nature since the interval where solution exists may be smaller than the original interval, where f(t, x) is defined.

Note 2: The conditions of existence and uniqueness theorem are sufficient but not necessary.

Note 3: Any function f(x) = mx + b, m ≠ 0 must be a Lipschitz function. If f is Lipschiz on R, then f is continuous with respect to x on R.

Ex 1.3.1 Show that the IVP $\frac{dx}{dt} = x$, x(0) = 1 has unique solution.

Solution: Here, f(t,x) = x, and $\frac{\partial f}{\partial x} = 1$. Clearly both of these exist and bounded around (0,1). Hence, unique solution exists and given by $x = e^t$.

Ex 1.3.2 Show that the IVP $\frac{dx}{dt} = 3x^{\frac{2}{3}}$, x(0) = 0 has no unique solution.

Solution: Here, $f\{t,x) = 3x^{\frac{2}{3}}$ is continuous on t – x plane.

Now,

$$\frac{|f(0,x_1) - f(0,0)|}{|x_1 - 0|} = 3\frac{|x_1^{\frac{2}{3}}|}{|x_1|} = \frac{3}{|x_1^{\frac{1}{3}}|}.$$

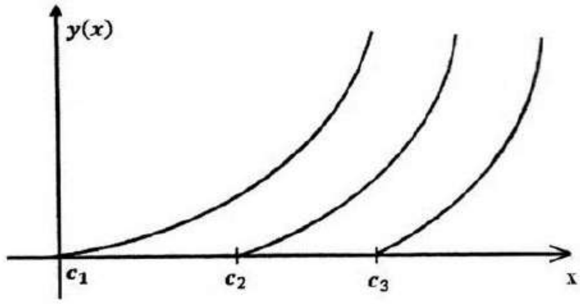


Figure 1.1 :

If we take x_1 very close to zero, $\frac{3}{|x_1^{\frac{1}{3}}|}$ becomes unbounded. Hence, Lipschitz condition does not hold around a region about (0, 0). So, we can not say anything about uniqueness of solution. Since Lipschitz condition is a sufficient condition.

Now, $$\int \frac{dx}{x^{\frac{2}{3}}} = \int dt,$$

$$x^{\frac{1}{3}} = t + c.$$

At, $t = 0,\ x = 0 \Rightarrow C = 0.$ (1.2)

So, $x = t^3$ is a solution. Again $x = 0$ is also solution. Hence, the solution is not unique. The solution can be written as $x(t) = \begin{cases} (t-c)^2 & x \geq c \\ O & x \geq c. \end{cases}$

Ex. 1.3.3 Show that the function $f(t,x) = tx^{\frac{1}{3}}$ does not satisfy Lipschitz condition in any domain containing origin but IVP $\frac{dx}{dt} = tx^{\frac{1}{3}}$, $x(0) = 0$ has a unique solution.

Solution: Now, $\frac{|f(t, x_1) - f(t, 0)|}{|x_1 - 0|} = \frac{tx_1^{\frac{2}{3}}}{x_1} = \frac{t}{x_1^{\frac{2}{3}}}$, which is unbounded in any domain containing the origin. Hence, the function f(t,x) does not satisfy Lipschitz's condition.

Again, we have

$$\frac{dx}{dt} = tx^{\frac{1}{3}}$$

$$\Rightarrow \quad \int \frac{dx}{x^{\frac{1}{3}}} = \int t\,dt$$

$$\Rightarrow \quad 3x^{\frac{2}{3}} = t^2 + c$$

At $x(0) = 0 \Rightarrow c = 0.$

So, the solution is $3x^{\frac{2}{3}} = t^2$. This is an example in which Lipschitz condition is only a sufficient condition for uniqueness and not a necessary one.

Ex 1.3.4 Discuss the existence and uniqueness for the IVP $\frac{dx}{dt} = \frac{2x}{t}$, $x(0) = x_0$.

Solution: Here, $f(t,x) = \frac{2x}{t}$, and $\frac{\partial f}{\partial x} = \frac{2}{t}$. Clearly both of these are unbounded around $(0, x_0)$. So, nothing can be said from the existence and uniqueness theorem.

If we solve the equation we find $x = At^2$. When $x_0 \neq 0$, there exists no solution of this differential equation. If $x_0 = 0$, then we have infinite number of solutions $x = kt^2$ (k any real constant). This typical behavior arises due to singular nature of ODE at $t = 0$.

Wellposedness

The initial value problem (IVP) $\frac{dx}{dt} = f(t, x)$ with $x(t_0) = x_0$ is said to be wellposed if

(i) the system has a solution in a class of functions (existence of solution)

(ii) the solution is unique in class of function (uniqueness of solution)

(iii) the solution depends continuously on initial condition x_0 (stability of solution).

1.4 Picard iteration for IVP (Initial Value Problem)

This method gives approximate solution of IVP (1.1). Note that the IVP (1.1) is equivalent to the integral equation $x(t) = x_0 + \int_{t_0}^{t} f(s, x(t))ds$. A rough approximation to the solution $x = x(t)$ is given by the function $x_0(t) = x_0$, which is a simply horizontal line through (t_0, x_0). If we find

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(t))ds,$$

then $x_1(t)$ may be a little more closer to $x(t)$ than $x_0(t)$. In similar manner, we can find $x_2(t)$, $x_3(t)$ and so on. At the n-th stage

$$x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(t))ds.$$

The Picard theorem states that $x_n(t) \to x(t)$ as $n \to \infty$ giving the unique solution of the initial value problem.

THEOREM 1.4.1 If the function f(t,x) satisfy the existence and uniqueness theorem for IVP (1.1), the the successive approximation $x_n(t)$ converges to the unique solution x{t) of the IVP (1.1).

Ex 1.4.1 Find the first three approximation of the IVP $\frac{dx}{dt} = 1 + tx,\ x(0) = 1$.

Solution: Let the zeroth approximation be $x_0(t) = x(0) = 1$.

The successive approximation are as follows :

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(t))ds,$$

$$\Rightarrow \quad x_1(t) = 1 + \int_0^t t(1+s)ds = 1 + t + \frac{t^2}{2}.$$

Again, $$x_2(t) = 1 + \int_0^t \left[1 + s\left(1 + s + \frac{s^2}{2}\right)\right] ds$$

$$\Rightarrow \quad x_2(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8}.$$

Also, $$x_3(t) = 1 + \int_0^t [1 + s x_2(s)]ds$$

$$\Rightarrow \quad x_3(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8} + \frac{t^5}{15} + \frac{t^6}{48}.$$

Ex 1.4.2 Consider the IVP $\frac{dx}{dt} = x^{\frac{1}{2}}$, $x(0) = 0$. Show that the uniqueness of solution in the Picard's theorem may fails, when the Lipschitz condition is dropped.

Solution: $f(t,x) = x^{\frac{1}{2}}$ is continuous for all values of x.

So, $\frac{|f(t, x_1) - f(t, 0)|}{|x_1 - 0|} = \frac{x_1^{\frac{1}{2}}}{x_1} = \frac{1}{x_1^{\frac{1}{2}}}$ which is unbounded as $x_1 \to 0$. So f(t,x) does not satisfy Lipschitz condition in the neighbourhood of (0,0).

The Picard's successive approximations at (n+1)th step are $x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(x, x_n(t))ds$.

Here, $x(0) = 0$, $x_0(t) = 0$, $x_1(t) = 0 + \int_0^t f(s, x_0(s))ds = 0$. Similarly, other approximations are zero. Hence, $x_n(t) \to 0$ as $n \to \infty$ and so $x_n(t) \equiv 0$ is a solution of the IVP.

Again, $\frac{dx}{\sqrt{x}} = dt \Rightarrow 2\sqrt{x} = t + c$, at $t = 0, x = 0 \Rightarrow c = 0$. Thus $x = \frac{t^2}{4}$ is the solution of IVP. This solution is different from the solution $x(t) = 0$ as obtained from Picard's theorem. Hence proved.

1.5 Single Species Growth Equation

Let consider Malthusian population growth model equation $\frac{dx}{dt} = ax$.

Here $x = x(t)$ is an unknown real-valued function of a real variable t. Also, a is a parameter; for each value of a we have a different differential equation. The solutions of this equation are obtained from calculus: If k is any real number, then the function $x(t) = ke^{at}$ is a solution.

Suppose that a function x(t) satisfying the differential equation is also required to satisfy $x(t_0) = u_0$. Then we must have $ke^{at_0} = u_0$, so that $k = u_0e^{-at_0}$. Then the solution of the system is $x(t) = u_0e^{a(t-t_0)}$ Note that there is a special solution of this differential equation when $U_0 = 0 \Rightarrow k = 0$. This is the constant solution $x(t) \equiv 0$. A constant solution such as this is called an equilibrium solution or equilibrium point for the equation. Equilibria are often the most important solutions of differential equations. Equilibrium solution is obtained by setting $\frac{dx}{dt} = 0$.

The solution of system changes qualitatively with variation of a. If $a < 0$, $\lim_{t\to\infty} ke^{at} = 0$; if $a = 0$ ke^{at} =constant; if $a > 0$ $\lim_{t\to\infty} ke^{at} = \infty$ when $k > 0$, and equals $-\infty$ when $k < 0$. Note that the behaviour of solutions is quite different when a is positive and negative. When $a > 0$, all non-zero solutions tend away from the equilibrium point at 0 as t increases, whereas when $a < 0$, solutions tend toward the equilibrium point. The set of trajectories may imagine as a flow on R or R^2 or R^3. We say that the equilibrium point is a sourceunstable when nearby solutions tend away from it. The equilibrium point is a sink stable when nearby solutions tend toward it.

The above equation can be considered a simple model of population growth when $a > 0$. The quantity $x(t)$ measures the population of some species at time t. The assumption that leads to the differential equation is that the rate of growth of the population (namely, dx/dt) is directly proportional to the size of the population. Of course, this naive assumption omits many circumstances that govern actual population growth, including, for example, the fact that actual populations cannot increase without bound.

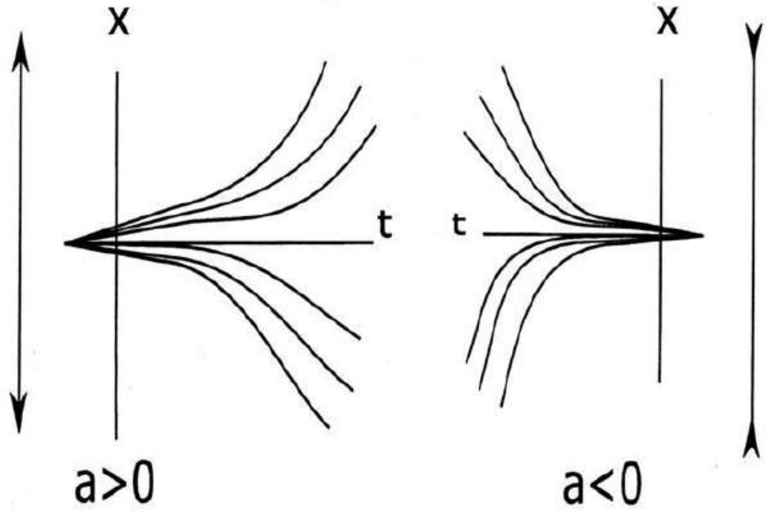


Figure. 1.2:

1.6 Logistic Population Model

In order to make restriction on unbounded growth of population, we can make the following further assumptions about the population model:

1. If the population is small, the growth rate is nearly directly proportional to the size of the population;

2. but if the population grows too large, the growth rate becomes negative. One differential equation that satisfies these assumptions is the logistic population growth model. This differential equation is

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right).$$

Here a and N are positive parameters: a gives the rate of population growth when x is small, while N represents a sort of "ideal" population or "carrying capacity."

Without loss of generality we will assume that N=1. That is, we will choose units so that the carrying capacity is exactly 1 unit of population, and $x(t)$ therefore represents the fraction of the ideal population present at time t. The logistic equation becomes

$$\frac{dx}{dt} = ax(1-x).$$

This is an example of a first-order, autonomous, nonlinear differential equation.

Now,

$$\int \frac{dx}{x(1-x)} = \int adt,$$

$$\Rightarrow \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = a \int dt,$$

$$\Rightarrow x(t) = \frac{ke^{at}}{1+ke^{at}}$$

Evaluating this expression at t = 0, $x(0) = x_0 \Rightarrow k = \frac{x_0}{1-x_0}$ and solution is

$$x(t) = \frac{x_0 e^{at}}{1-x_0+x_0 e^{at}}.$$

So this solution is valid for any initial population x_0. When x(0) = 1, we have an equilibrium solution, since x(t) reduces to $x(t) \equiv 1$. Similarly, $x(t) \equiv 0$ is also an equilibrium solution.

1.7 Single Species Model with Harvesting

Now let's modify the logistic model to take into account harvesting of the population. Suppose that the population obeys the logistic assumptions with the parameter a=1, but is also harvested at the constant rate h. The differential equation becomes $\frac{dx}{dt} = x(1-x) - h$ where $h \geq 0$ is a new parameter.

In Figure 1.3, we display the graph of f(x) in three different cases: $0 < h < 1/4$, $h = 1/4$, and $h > 1/4$.

It is straightforward to check that f(x) has two roots when $0 \leq h < 1/4$, one root when $h = 1/4$, and no roots if $h > 1/4$, as illustrated in the graphs.

As a consequence, the differential equation has two equilibrium points x_1 and x_2 with $0 \leq x_1 < x_2$ when $0 < h < 1/4$.

As h passes through h = 1/4, we encounter example of a bifurcation. The two equilibria x_1 and x_2 coalesce as h increases through 1/4 and then disappear when h > 1/4. Moreover, when h > 1/4, we have f(x) < 0 for all x. Mathematically, this means that all solutions of the differential equation decreases to $-\infty$ as time goes on.

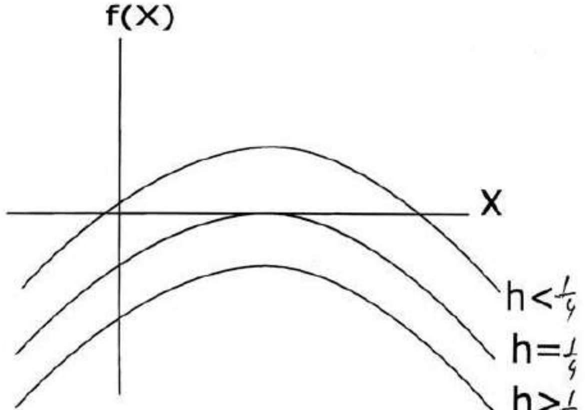


Figure: 1.3

1.8 Dynamical System

Dynamics is a time-evolutionary process. It may be deterministic or stochastic.

A system of n first order differential equation is called dynamical system of dimension n which determines the time behaviour of evolutionary system. The subject of dynamical systems concerns the evolution of systems in time. In continuous time, the systems may be modeled by ordinary differential equations (ODEs), partial differential equations (PDEs), or other types of equations (e.g., integro-differential or delay equations); in discrete time, they may be modeled by difference equations or iterated maps.

A dynamical system is described by two things: a state and a dynamics. The state of a dynamical system is the values of all the variables that describe the system at a particular time instant. And by dynamics of the system it is meant that the set of laws or equations that describe how the state of the system evolves with time. Usually this set of equations consists of a system of coupled differential equations, one for each of the states variables. Each state of a system can be represented by a point in a space, called state space or phase space. The dynamics of a system is governed by the line connecting the consecutive points in state space which is called the trajectory of the system.

The mathematical form for the dynamical system is written as

$$\frac{dx}{dt} = f(t, x), \text{ .... } (1.3)$$

where x(t) is a n- vector and $f(t, x)$ is sufficient smooth function defined on some subset $U \subset R^n \times R$. Dynamical system is linear and nonlinear according as f(t,X) is linear or nonlinear. The most common form of dynamical system is Newton's second law of motion: $\dot{x} = y$, $\dot{y} = F$, where F is force on unit mass particle.

If the right hand side of (1.3) is explicitly time independent then the system is called autonomous and the trajectories of such systems do not change as time goes on. On the other hand, if the right hand side of (1.3) is explicitly time dependent then the system is called non-autonomous. Logistic equation: $\dot{x} = ax(1-x)$ is example of autorpn^ous, system. Harmonic oscillator is example of nonautonomous system represented by $m\ddot{x} + b\dot{x} + kx = F$ cost.

Let $P \subset R^m$, $m \in N$, $\bar{x} = (x_1, \dots x_m) \in P$, $t \in R$. A continuous dynamical system is defined as $\dot{\bar{x}} = F(\bar{x})$, where $F : P \to P$, $F = (f_1\ f_2 \dots f_m)$. The swinging of a pendulum is governed by the equation

$$\ddot{x} + \frac{g}{L}\sin x = 0,$$

where x is the angle of the pendulum from vertical, g is the acceleration due to gravity, and L is the length of the pendulum. The equivalent system is nonlinear dynamical system.

Let, $P \subset R^m$, $m \in N$; $x_n \in P$, $n \in Z$. Then $x_{n+i} = G(x_n)$, where $G : P \to P$, is a discrete dynamical system (or discrete-time dynamical system) and $G = (g_1, g_2, \dots g_m)$. Logistic map, Henon map are two very famous discrete dynamical system.

1.9 Summary

In this unit we provided theorems to understand the basic concepts of existence and uniqueness of solution of ordinary differential equation. Examples are given in order to clarify more on this topic. Some basic ideas of continuous and discrete dynamical system are provided. Finally, single species growth equation, logistic equation, and single species model with harvesting are also discussed.

1.10 Keywords

Dynamical system, Existence and uniqueness of differential equation, Picard's successive approximations, Growth equation, Logistic equation.

1.11 Further Reading

1. Coddington, E. A. (2012). An introduction to ordinary differential equations. Courier Corporation.

2. Simmons, G. F. (2016). Differential equations with applications and historical notes. CRC Press.

3. Poria, S., k Dhiman, A. (2017). Existence and uniqueness of solution to ODEs: Lipschitz continuity. Resonance, 22(05).

4. Perko, L. (2013). Differential equations and dynamical systems (Vol. 7). Springer Science k Business Media.

5. Hirsch, M. W., Devaney, R. L., k Smale, S. (1974). Differential equations, dynamical systems, and linear algebra (Vol. 60). Academic press.

6. Jordan, D. W., K Smith, P. (1999). Nonlinear ordinary differential equations: an introduction to dynamical systems (Vol. 2). Oxford University Press, USA.

7. Strogatz, S. H. (2018). Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering. CRC Press.

8. Layek, G. C. (2015). An Introduction to Dynamical Systems and Chaos, Springer.

1.12 Exercise

1. (A general example of non-uniqueness) Consider the initial value problem $x = |x|^{\frac{p}{q}}, x(0) = 0$, where p and q are positive integers with no common factors.

   a) Show that there are an infinite number of solutions if $p < q$.

   b) Show that there is a unique solution if $p > q$.

2. Discuss existence and uniqueness of logistic equation $\dot{x} = rx(1 - x)$, $x(0) = x_0$ for different values of x_0.

3. Find the first three successive approximations $x_1(t)$, $x_2(t)$ and $x_3(t)$ for the initial value problem $\dot{x} = x^2$, $x(0) = 1$. Also, use mathematical induction to show that for all $n \geq 1$, $x_n(t) = 1 + t + ... + t^n + 0(t^{n+1})$ as $t \to 0$.

4. Find the general solution of the logistic differential equation with constant harvesting $\frac{dx}{dt} = x(1 - x) - h$ for all values of the parameter $h > 0$.

5. Consider the nonautonomous differential equation $\dot{x} = \begin{cases} x-4 & \text{if } t<5 \\ 2-x & \text{if } t\geq 5. \end{cases}$

(a) Find a solution of this equation satisfying $x(0) = 4$. Describe the qualitative behavior of this solution.

(b) Find a solution of this equation satisfying $x(0) = 3$. Describe the qualitative behavior of this solution.

(c) Describe the qualitative behavior of any solution of this system as $t \to \infty$.

6. (Tumor growth) The growth of cancerous tumors can be modeled by the Gompertz law $\dot{N} = -aN \ln(bN)$, where N(t) is proportional to the number of cells in the tumor, and $a, b > 0$ are parameters.

a) Interpret a and b biologically.

b) Sketch the vector field and then graph N(t) for various initial values.

7. (The Allee effect) For certain species of organisms, the effective growth rate $\frac{\dot{N}}{N}$ is highest at intermediate N. This is the called the Allee effect (Edelstein-Keshet 1988). For example, imagine that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.

a) Show that $\frac{\dot{N}}{N} = r - a(N - b)^2$ provides an example of Allee effect, if r, a, and b satisfy certain constraints, to be determined.

b) Find all the fixed points of the system and classify their stability.

c) Sketch the solutions N(t) for different initial conditions.

d) Compare the solutions N(t) to those found for the logistic equation. What are the qualitative differences, if any?

Unit 2 ❑ Planar Linear Systems

Structure

2.0 Objective

In this unit we begin with the simplest class of higher dimensional systems, namely, linear systems in two-dimension. These systems are interesting in their own way, and, as we'll see later, they also play important role in classification of fixed points of nonlinear systems.

2.1 Introduction

A system of differential equations is a collection of n interrelated differential equations of the form

$$\begin{aligned}\dot{x}_1 &= f_1(t,x_1,x_2,\ldots,x_n),\\ \dot{x}_2 &= f_2(t,x_1,x_2,\ldots,x_n),\\ &\vdots\\ \dot{x}_n &= f_n(t,x_1,x_2,\ldots,x_n),\end{aligned} \tag{2.1}$$

Here the functions f_j are real-valued functions of the n + 1 variables $x_1, x_2, \ldots, x_n$, and t. Unless otherwise specified, we will always assume that the f_j are C^∞ functions. This means that the partial derivatives of all orders of the f_j exist and are continuous. In vector notation, $\dot{X} = F(t,X)$. A solution of this system is then a function of the form $X(t) = (x_1(t),\ldots, x_n(t))$ that satisfies .the equation, so that $\dot{X}(t) = F(t,X(t))$ where $\dot{X}(t) = (\dot{x}_1(t), \ldots,\dot{x}_n(t))$. n = 2 gives the planer system.

A vector X_0 for which $F(X_0) = 0$ is called an equilibrium point for the system. An equilibrium point corresponds to a constant solution $X(t) \equiv X_0$ of the system.

2.2 Planar System

A two-dimensional linear system is a system in the form

$$\dot{x} = ax + by$$

$$\dot{y} = cx + dy \qquad (2.2)$$

where a, b, c,and d are real parameters. The two variables x, y might represent, for example, two interacting animal species in an ecological system, two different conductances of ion channels in a cell membrane, two different chemicals in a chemical reaction, or the concentration of a drug in two different organs.

This system (2.2) can be written more compactly in matrix form as $\dot{X} = AX$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix} \in R^2$. Since the solution of linear homogeneous system forms a vector space hence $c_1X_1 + c_2X_2$ is also a solution whenever X_1 and X_2 are solution of the equation (2.2).

The solutions of $\dot{X} = AX$, can be visualized as trajectories moving on (x, y) plane, in this context is called the phase plane. Also notice that $\dot{X} = 0$ when $X = 0$, so $X^* = 0$ is always a fixed point for any choice of A.

Proposition 2.2.1 The planar linear system $\dot{X} = AX$ has

1. A unique equilibrium point (0,0) if $\det A \neq 0$.

2. A straight line of equilibrium points if $\det A = 0$ (and A is not the 0 matrix).

Theorem 2.2.1 Suppose that V_0 is an eigenvector for the matrix A with associated eigenvalue λ. Then the function $X(t) = e^{\lambda t}V_0$ is a solution of the system $\dot{X} = AX$.

Proof: Suppose V_0 is a nonzero vector for which we have $AV_0 = \lambda V_0$ where $\lambda \in R$.

To prove $X(t) = e^{\lambda t}V_0$ is a solution of $\dot{X} = AX$.

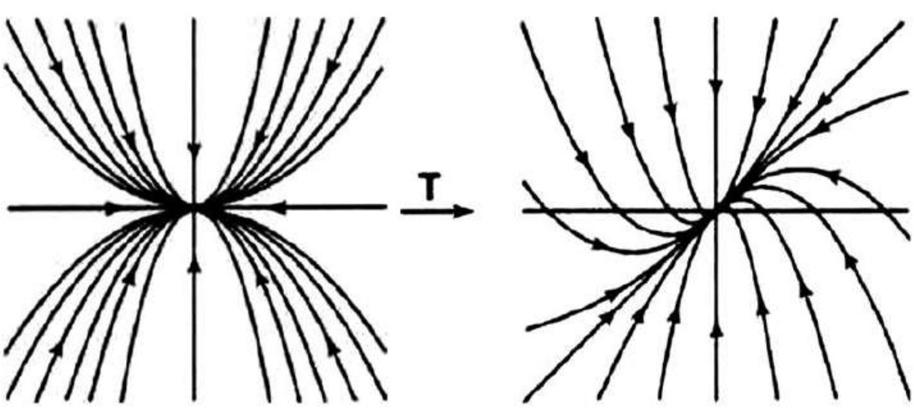
$$\begin{aligned}\dot{X} &= \lambda e^{\lambda t}V_0\\ &= e^{\lambda t}(\lambda V_0)\\ &= e^{\lambda t}(AV_0)\\ &= A(e^{\lambda t}V_0)\\ &= AX(t)\end{aligned}$$

So X(t) does indeed solve the system of equations.

2.3 Changing Coordinates

Any 2×2 matrix that is in one of the following three forms is said to be in canonical form, $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}, \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}, \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}$, λ may equal μ in the first case, Given any linear system $\dot{X} = AX$, we can always "change coordinates" so that the new system's coefficient matrix is in canonical form and hence easily solved.

Now, instead of considering a linear system $\dot{X} = AX$, suppose we consider a different system $\dot{Y} = (T^{-1}AT)Y$ for some invertible matrix T ($\det T \neq 0$). Note that if Y(t) is a solution of this new system, then $X(t) = TY(t)$ solves $\dot{X} = AX$. We can always change the coordinates by finding a suitable matrix that converts a given linear system to one of the canonical forms.



Study of non-diagonalizable system is beyond syllabus. It requires concepts of Jordan canonical forms and generalized eigen vectors.

2.4 Solving Linear System

We can find the solution to equation (2.2) in terms of a, b, c, and d by solving the characteristic equation of A, given by det(A – λI) = 0, where I is the identity matrix.

Thus the general solution is $X(t) = C_1e^{\lambda_1 t}V_1 + c_2e^{\lambda_2 t}V_2$, where V_1 and V_2 are linearly independent eigenvectors corresponding to eigenvalue λ_1 and λ_2, respectively. In particular, any initial condition x_0 can be written as linear combination of eigenvectors as $X_0 = c_1V_1 + c_2V_2$. Where C_1, C_2 are consta.

THEOREM 2.4.1 Suppose A has a pair of real eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors V_1 and V_2. Then the general solution of the linear system $\dot{X} = AX$ is given by $X(t) = C_1e^{\lambda_1 t}V_1 + C_2e^{\lambda_2 t}V_2$.

THEOREM 2.4.2 Suppose A has a pair repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$ and associated with one linearly independent eigenvector V_1. Let V_2 be generalized eigen vector of A $[(A - \lambda I)V_2 = V_1]$. Then the general solution of the linear system $\dot{X} = AX$ is given by $X(t) = c_1e^{\lambda t}V_1 + c_2(te^{\lambda t}V_1 + e^{\lambda t}V_2)$.

THEOREM 2.4.3 Suppose A has a pair complex conjugate eigenvalues $\alpha \pm i\beta$ and associated with eigenvectors $V_1 + iV_2$ and $V_1 - iV_2$. Then the general solution of the linear system $\dot{X} = AX$ is given by $X(t) = C_1e^{\alpha t}(V_1 \cos \beta t - V_2 \sin \beta t) + c_2e^{\alpha t}(V_1 \sin \beta t + V_2 \cos \beta t)$.

Ex 2.4.1 Find the general solution of the linear system $\dot{X} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} X$.

Solution: The characteristic equation is $\lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1) = 0$, so the system has eigenvalues –1 and –2. The eigenvector corresponding to the eigenvalue –1 is given by solving the equation $\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ one independent equation $x + y = 0$. Hence one eigenvector associated to the eigenvalue –1 is $(1, -1)^t$. In similar fashion we compute that an eigenvector associated to the eigenvalue –2 as $(1, -2)^t$. Note that these two eigenvectors are linearly independent. Therefore, by the previous theorem, the general solution of this system is $X(t) = c_1e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Ex 2.4.2 Find the general solution of the linear system $\dot{x} = 3x - 4y$, $\dot{y} = x - \dot{y}$.

Solution: The characteristic equation is $\begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1, 1$.

The matrix has repeated eigen values. Let $V = (v_1, V_2)^t$ be a eigen vector corresponding to $\lambda = 1$.

$(A - I)V = 0 \Rightarrow \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ $V_1 - 2v_2 = 0$. So, $V = (2,1)^t$ be the only eigen vector corresponding to $\lambda = 1$.

Let U be the generalized eigen vector corresponding to $\lambda - 1$, so $(A - I)U = V$, which implies $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow u_1 - 2u_2 = 1$. So, $U = (1,0)^t$ is another independent eigen vector corresponding to $\lambda = 1$. Hence required general solution is $X(t) = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^t \left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

Ex 2.4.3 Solve the homogeneous system $\dot{X} = AX$, where $\begin{pmatrix} 3 & 2 \\ -5 & 1 \end{pmatrix}$.

Solution: The characteristic equation is $\lambda^2 - 4\lambda + 13 = 0 \Rightarrow \lambda = \dfrac{4 \pm \sqrt{16-52}}{2} \Rightarrow \lambda = 2 \pm 3i$. A has complex eigen values. The eigen vectors V_1 corresponding to $2 + 3i$ is obtained from

$$(A - \lambda I)V = 0,$$

$$\Rightarrow \begin{pmatrix} 3-2-3i & 2 \\ -5 & 1-2-3i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\Rightarrow (1 - 3i)v_1 + 2v_2 = 0 - 5v_1 + (-1 - 3i)v_2 = 0$$

A non-trivial solution of the system is $v_1 = 2, v_2 = -1 + 3i$, so $V_1 \begin{pmatrix} 2 \\ -1+3i \end{pmatrix}$.

Similarly, the other eigen vector V_2 corresponding to $2 - 3i$ is $\begin{pmatrix} 2 \\ -1-3i \end{pmatrix}$.

The general solution is given by $X(t) = c_1 S_1 + c_2 S_2$, where

$$S_1 = e^{2t}\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}\cos 3t - \begin{pmatrix} 0 \\ 3 \end{pmatrix}\sin 3t\right\},$$

$$S_2 = e^{2t}\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}\sin 3t + \begin{pmatrix} 0 \\ 3 \end{pmatrix}\cos 3t\right\}.$$

2.5 Phase Portrait

A qualitative understanding of two-dimensional nonlinear systems can often be gained from studying the phase plane of the system. This can provide information about multiple stable and unstable fixed points that is not given by numerical integration. By flowing along the vector field, a phase point traces out a solution x(t), corresponding to a trajectory winding through the phase plane. Furthermore, the entire phase plane is filled with trajectories, since each point can play the role of an initial condition. For nonlinear system, there's typically no hope of finding the trajectories analytically.

To sketch the phase portrait, it is helpful to plot nullclines, defined as the curves where either $\dot{x} = 0$ or $\dot{y} = 0$. The nullclines indicate where the flow is purely horizontal or vertical.

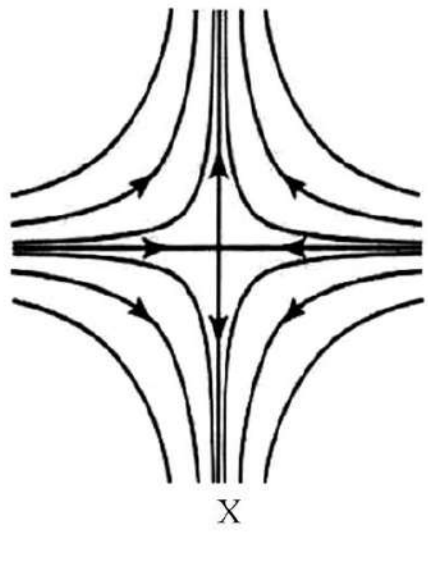
Real Distinct Eigenvalues

Consider $X' = AX$ and suppose that A has two real eigenvalues $\lambda_1 < \lambda_2$.

Case 1: $\lambda_1 < 0 < \lambda_2$

Consider $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then this can be solved immediately since the system decouples into two unrelated first-order equations: $x' = \lambda_1 x$, $y' = \lambda_2 y$. So λ_1 and λ_1 are the eigenvalues of A. An eigenvector corresponding to λ_1 is (1,0) and to λ_2 is (0,1). Hence we find the general solution $X(t) = c_1 e^{\lambda_1 t} \binom{1}{0} + c_2 e^{\lambda_2 t} \binom{0}{1}$.

Also, $\frac{dy}{dx} = \frac{\lambda_2}{\lambda_1}\frac{y}{x}$ is the differential equation of the phase paths. This gives $y = kx^{\frac{\lambda_2}{\lambda_1}}$, k is integrating constant.



Since $\lambda_1 < 0$, the straight-line solutions of the form $c_1 e^{\lambda_1 t}(1, 0)^t$ lie on the x-axis and tend to (0,0) as $t \to \infty$. This axis is called the stable line. Since $\lambda_2 > 0$, the solutions $c_2 e^{\lambda_2 t}(0,1)^t$ lie on the y-axis and tend away from (0,0) as $t \to \infty$; this axis is the unstable line.

All other solutions (with $c_1, c_2 \neq 0$) tend to ∞ in the direction of the unstable line, as $t \to \infty$, since X(t) comes closer and closer to $(0, c_2 e^{\lambda_2 t})$ as t increases. In backward time, these solutions tend to ∞ in the direction of the stable line.

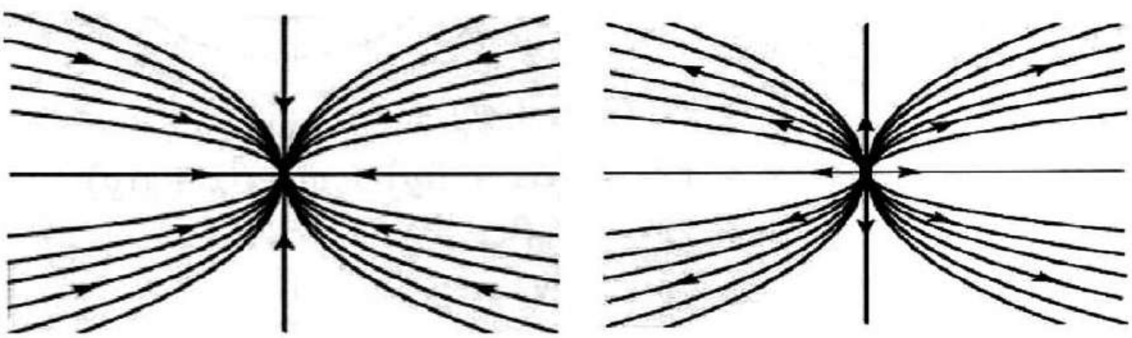
This kind of equilibrium point at the origin is called saddle. A saddle is always unstable.

Case 2: $\lambda_1 < \lambda_2 < 0$

[illegible] $_1e^{\lambda_1 t}\binom{1}{0} + c_2e^{\lambda_2 t}\binom{0}{1}$. Unlike the saddle case, now all solutions tend to (0,0) as $t \to \infty$.

We compute the slope dy/dx of a solution with $c_2 \neq 0$. We write $x(t) = c_1e^{\lambda_1 t}$, $y(t) = c_2e^{\lambda_2 t}$ and compute $\dfrac{dy}{dx} = \dfrac{\lambda_2 c_2 e^{\lambda_2 t}}{\lambda_1 c_1 e^{\lambda_1 t}} = \dfrac{\lambda_2 c_2}{\lambda_1 c_1} e^{(\lambda_2 - \lambda_1)t}$.

Since $\lambda_2 - \lambda_1 > 0$, it follows that these slopes approach $\pm\infty$ (provided $c_2 \neq 0$). Thus these solutions tend to the origin tangentially to the y-axis. Since $\lambda_1 < \lambda_2 < 0$, we call λ_1 the stronger eigenvalue and λ_2 the weaker eigenvalue. In this case the equilibrium point is called stable node or a sink. Figure 2.1 (a) is phase portrait of a sink.



(a) **(b)**

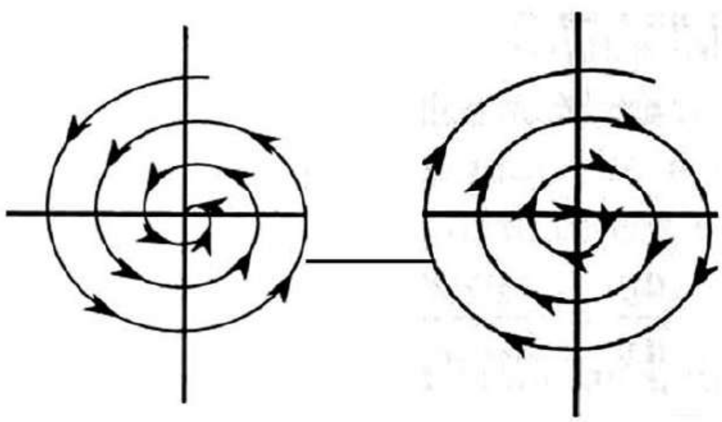
Figure 2.1:

Case 3: $0 < \lambda_2 < \lambda_1$

The general solution and phase portrait remain the same, except that all solutions now tend away from(0, 0) along the same paths. Figure 2.1(b). This is the case of unstable node.

Complex Eigenvalues

Case I: Consider $X' = AX$ where $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, and $\alpha, \beta \neq 0$. The characteristic equation is now $\lambda_2 - 2\alpha\lambda + \alpha^2 + \beta^2$, so the eigenvalues are $\lambda = \alpha \pm i\beta$. The eigen vector corresponding to $\alpha + i\beta$ is $(1,i)^t$. The general solution is $X(t) = c_1e^{\alpha t}X_{real}(t) + c_2e^{at}X_{imaginary}(t)$, where $X_{real}(t) = \begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}$ and $X_{imaginary}(t) = \begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}$. The $e^{\alpha t}$ term converts solutions into spirals that either spiral into the origin (when $\alpha < 0$) or away from the origin ($\alpha > 0$). In these cases the equilibrium point is called a spiral sink or spiral source, respectively. See Figure 2.2. The equilibrium point origin is called focus.



Spiral Sink Spiral Source

Figure 2.2:

Alternatively, let put $x = r\cos\theta$, $y = r\sin\theta$ in $X' = AX$. The system is transformed from $(x,y) \to (\pi,\theta)$.

$$\text{Now,} \quad r^2 = x^2 + y^2,$$

$$\Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$$

$$\Rightarrow r\dot{r} = x(\alpha x + \beta y) + y(-\beta x + \alpha y)$$

$$\Rightarrow r\dot{r} = \alpha(r^2 + y^2)$$

$$\Rightarrow r\dot{r} = \alpha r^2$$

$$\Rightarrow \dot{r} = \alpha r$$

$$\Rightarrow r(t) = k_1 e^{\alpha t},\ k_1 \text{ is constant.}$$

Again, $\tan\# = \dfrac{y}{x}$

$$\Rightarrow \sec 2\theta\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2} = \frac{x(-\beta x + \alpha y) - y(\alpha x + \beta y)}{x^2}$$

$$\Rightarrow \left(1 + \frac{y^2}{x^2}\right)\dot{\theta} = \frac{-\beta(x^2 + y^2}{x^2}$$

$$\Rightarrow r^2\dot{\theta} = -\beta r^2$$

$$\Rightarrow \dot{\theta} = \beta$$

$$\Rightarrow \theta(t) = -\beta t + k_2,\ k_2 \text{ is constant.}$$

Case II: Now put $\alpha = 0$ in the previous case and $A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$, and ($\beta \neq 0$. The characteristic polynomial is $\lambda^2 + \beta^2 = 0$, so the eigenvalues are now the imaginary numbers $\pm i\beta$.

We solve, $\begin{pmatrix} -i\beta & \beta \\ -\beta & -i\beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow i\beta x = \beta y$. The corresponding eigen vector is $(1,i)^t$.

The general solution is $X(t) = C_1 X_{real}(t) + c_2 X_{irnaginary}(t)$, where $X_{real}(t) = \begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}$ and $X_{imaginary}(t) = \begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}$.

Note that each of these solutions is a periodic function with period $2\pi/\beta$. Indeed, the phase portrait shows that all solutions lie on circles centered at the origin. These circles are traversed in the clockwise direction if $\beta > 0$, counter-clockwise if $\beta < 0$. See Figure 2.3(a). This type of system is called a center.

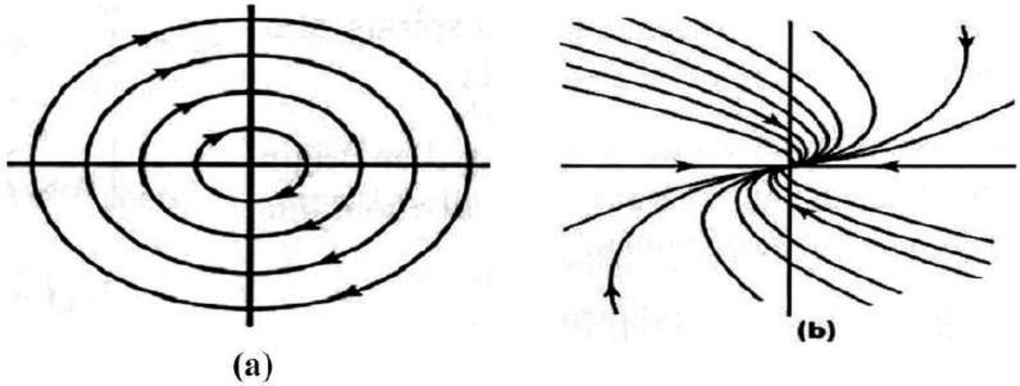


Figure 2.3:

Repeated Eigenvalues

Consider the case when $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ or $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. In the first case solutions are of the form $X(t) = c_1 e^{\alpha t} V$ (V is the eigen vector corresponding to λ). Each such solution lies on a straight line through (0,0) and either tends to (0,0) (if $\lambda < 0$, Figure 2.3(b)) or away from (0, 0) (if $\lambda > 0$).

In the second case, both eigenvalues are equal to λ, but now there is only one linearly independent eigenvector given by $(1,0)^t$. The solution of the system may be written $c_1 e^{\lambda t} \binom{1}{0} + c_2 e^{\lambda t} \binom{t}{1}$.

2.6 Trace-Determinant Plane

The characteristic equation is

$$\lambda^2 - \tau\lambda + \Delta = 0, \tag{2.4}$$

where, $\tau = a + d$, $\Delta = ad - bc$.

Then $\lambda_1 = \dfrac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}$ and $\lambda_2 = \dfrac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$ are two solutions of quadratic equation (2.4).

We can show the type and stability of all the different fixed points on a single diagram (Figure 2.4). The axes are the trace τ and the determinant Δ of the matrix

A. $\tau = \lambda_1 + \lambda_2$, $\Delta = \lambda_1\lambda_2$ and $D = \tau^2 - 4\Delta$. The parabola represented by $\tau^2 - 4\Delta = 0$ is the borderline between nodes and spirals, star nodes and degenerate nodes are on the parabola.

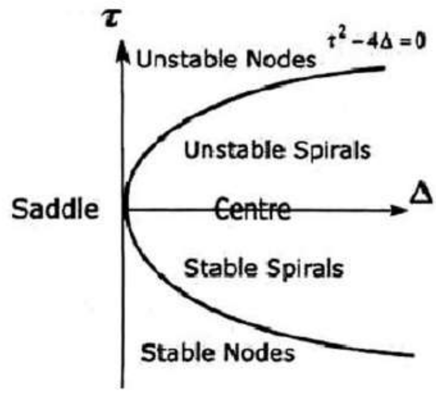


Figure 2.4:

If $\Delta = 0$, at least one of the eigen values is zero. The origin is not the only isolated fixed point. There is either a whole line of fixed points or a plane of fixed points.

The trace-determinant plane is a two-dimensional representation of what is really a four-dimensional space, since 2×2 matrices are determined by four parameters, the entries of the matrix. Thus there are infinitely many different matrices corresponding to each point in the TD-plane.

Nature	**Eigen values**	$\tau - \lambda_1 + \lambda_2$	$\Delta - \lambda_1\lambda_2$	$D = r^2 - 4\Delta$
Saddle	$\lambda_{1,2}$ are real distinct and opposite sign	–	$\Delta < 0$	$D > 0$
Stable Node	$\lambda_{1,2}$ are real distinct with negative real parts	$\tau < 0$	$\Delta > 0$	$D > 0$
Stable spiral	$\lambda_{1,2}$ are complex conjugate with negative real parts	$\tau < 0$	$\Delta > 0$	$D < 0$
Unstable node	$\lambda_{1,2}$ are real distinct with positive real parts	$\tau > 0$	$\Delta > 0$	$D > 0$
Unstable spiral	$\lambda_{1,2}$ are complex conjugate with positive real parts	$\tau > 0$	$\Delta > 0$	$D < 0$
Centre	$\lambda_{1,2}$ are complex conjugate and purely imaginary	$\tau = 0$	$\Delta > 0$	$D < 0$
Degenerate stable node	$\lambda_{1,2}$ are equal with negative part	$\tau < 0$	$\Delta > 0$	$D = 0$
Degenerate unstable node	$\lambda_{1,2}$ are equal with positive part	$\tau > 0$	$\Delta > 0$	$D = 0$

Ex 2.6.1 Draw the phase portrait for the system of initial value problem $x' = x + y$, $y' = 4x - 2y$, subject to the initial condition $(x_0, y_0) = (2, -3)$.

Solution: In matrix notation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic equation is the system

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0,$$

$$\Rightarrow \quad \lambda^2 + \lambda - 6 = 0,$$

$$\Rightarrow \quad \lambda_1 = 2,\ \lambda_2 = -3.$$

The eigenvector corresponding to the eigenvalue 2 is given by solving the equation $\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ one independent equation $\upsilon_1 - \upsilon_2 = 0$. Hence one eigenvector associated to the eigenvalue 2 is $(1,1)^t$. In similar fashion we compute that an eigenvector associated to the eigenvalue -3 is $(1, -4)^t$.

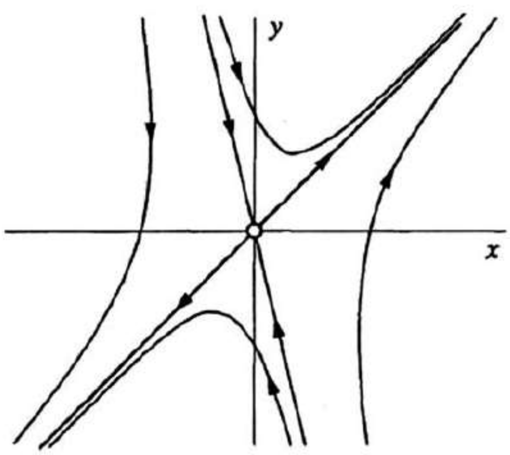


Figure 2.5:

Next we write the general solution as a linear combination of eigensolutions. The general solution is $X(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

Finally, we compute c_1 and c_2 to satisfy the initial condition $(x_0, y_0) = (2, -3)$. At $t = 0$, becomes

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$\Rightarrow 2 = c_1 + c_2,$$

$$-3 = c_1 - 4c_2$$

$$\Rightarrow c_1 = c_2 = 1.$$

Substituting yields,

$$x(t) = e^{2t} + e^{-3t},$$
$$y(t) = e^{2t} - 4e^{-3t}.$$

The eigen values are $\lambda_1 = 2, \lambda_2 = -3$. Hence the first eigensolution grows exponentially, and the second eigensolution decays. This means the origin is a saddle point. Its stable manifold is the line spanned by the eigenvector $\upsilon_2 = (1, -4)$, corresponding to the decaying eigensolution. Similarly, the unstable manifold is the line spanned by $\upsilon_1 = (1,1)$. As with all saddle points, a typical trajectory approaches the unstable manifold as $t \to \infty$, and the stable manifold as $t \to -\infty$. Figure 2.5 shows the phase portrait.

Ex 2.6.2 Sketch a typical phase portrait for the system whose coefficient matrix A = $\begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$.

Solution: The characteristic equation is the system is

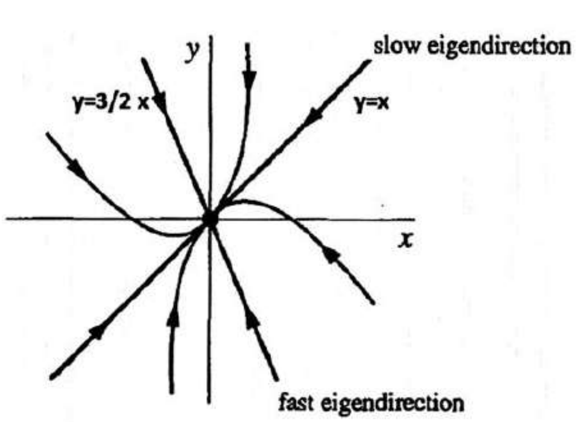
$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} -4-\lambda & 2 \\ -3 & 1-\lambda \end{vmatrix} = 0,$$

$$\Rightarrow \quad \lambda^2 + 3\lambda + 2 = 0,$$

$$\Rightarrow \quad \lambda_1 = -1, \lambda_2 = -2.$$

The eigen vectors corresponding to -1 is $(1,3/2)^t$ and -2 is $(1,1)^t$.



Both eigen values are less than zero, then both eigensolutions decay exponentially. The fixed point is a stable node. Trajectories typically approach the origin tangent to the slow eigendirection, defined as the direction spanned by the eigenvector with the smaller $|\lambda|$. In backwards time ($t \to -\infty$), the trajectories become parallel to the fast eigendirection.

2.7 Linearization near fixed points of two- dimensional systems

In this section we will discuss the linearization technique and approximate the phase portrait near a fixed point by that of a corresponding linear system. Now consider a system in R^2 as

$$\dot{x}_1 = f_1(x_1, x_2),$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

where f_1 and f_2 are given functions. Let (x^*,y^*) be the fixed point, i.e,

$$f_1(x^*, y^*) = 0, \qquad f_2(x^*y^*) = 0.$$

Let $u = x - x^*$, $v = y - y^*$ denote the components of a small disturbance from the fixed point. To see whether the disturbance grows or decays, we need to derive equations for u and v.

Now,

$$u = \dot{x} = f_1(x^* + u, y^* + v)$$

$$= u\frac{\partial f_1}{\partial x} + v\frac{\partial f_1}{\partial y} + O(u^2, v^2, uv)$$

$$= u\frac{\partial f_1}{\partial x} + v\frac{\partial f_1}{\partial y} + O(u^2, v^2, uv)$$

Since u and v are very small, the quadratic or higher order terms are extremely small. Similarly we find

$$\dot{v} = u\frac{\partial f_2}{\partial x} + v\frac{\partial f_2}{\partial y} + O(u^2, v^2, uv)$$

Hence the disturbance (u, v) evolves according to

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_{(x^*,y^*)} \begin{pmatrix} u \\ v \end{pmatrix} + 0(\text{higher oder terms}). \tag{2.5}$$

The matrix $A = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_{(x^*, y^*)}$ is the Jacobian matrix at the fixed point (x^*, y^*).

Since the quadratic terms are extremely small and we neglect them, we obtain the linearized system $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_{(x^*, y^*)} \begin{pmatrix} u \\ v \end{pmatrix}$ (2.6)

Ex 2.7.1 Classify the equilibrium point at (0,0) for the system $\dot{x} = e^{-x-3y} - 1$, $\dot{y} = -x(1-y^2)$.

Solution: The linear approximation about the point (0,0) is

$$\dot{x} = -x - 3y$$

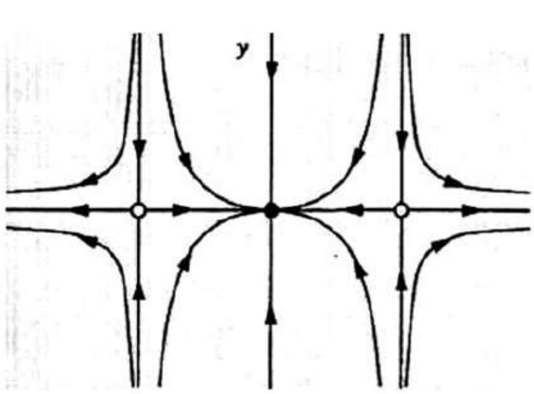
$$\dot{y} = -x.$$

The origin is the equilibrium point of the system.

$A = \begin{pmatrix} -1 & -3 \\ -1 & 0 \end{pmatrix}$, so $\tau = -1$, $\Delta = -3 < 0$, and $D = 13 > 0$. Hence the equilibrium point is a saddle point.

Ex 2.7.2 Find all the fixed points of the system $\dot{x} = -x + x^3$, $\dot{y} = -2y$, and use linearlization to classify them. Draw the phase portrait for the nonlinear system.

Solution: The fixed points will be obtained by solving $\dot{x} = 0$ and $\dot{y} = 0$. There are three fixed points : (0,0), (1,0) and (–1,0). The Jacobian matrix at A general point (x,y) $A = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} -1+3x^2 & 0 \\ 0 & -2 \end{pmatrix}$.



Next we evaluate A at the fixed points. At (0,0), $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, so (0, 0) is a stable node. At (±1,0), $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, so both (1,0) and (–1,0) are saddle points.

Since x and y equations are uncoupled; the system is essentially two independent first-order systems at right angles to each other. In the y-direction, all trajectories decay exponentially to y = 0. In the y-direction, trajectories are attracted to x = 0 and repelled from x = ±1.

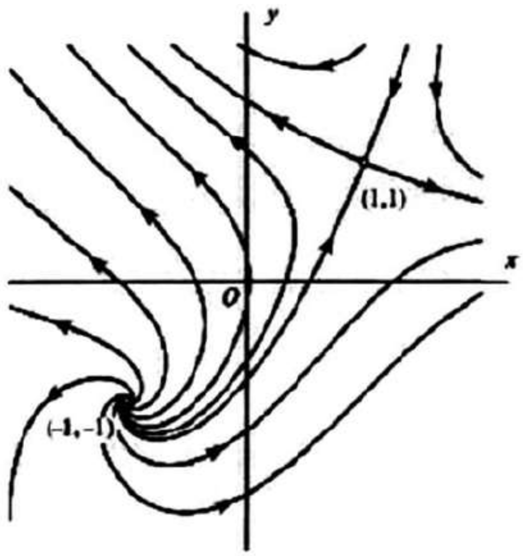
Ex 2.7.3 Find and classify the equilibrium points of $x' = x - y$, $y' = 1 - xy$. Finally sketch the phase diagram of the system.

Solution: The equilibrium points are at (–1,–1) and (1,1). The matrix for linearization, to be evaluated at each equilibrium point in turn, is $A = \begin{pmatrix} 1 & -1 \\ -y & -x \end{pmatrix}$.

At (–1,–1), by the transformation $u = x + 1, v = y + 1$ original system becomes (see equation 2.6)

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{2.7}$$

The characteristic equation of A becomes $\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = 1 \pm i$. Then (–1,–1) is unstable spiral. To obtain the direction of rotation, it is sufficient to use the linear equations (2.7) (or the original equations may be used): putting v = 0, u > 0 we find $\dot{v} = u > 0$, indicating that the rotation is counterclockwise as before.



At (1,1), we find that the eigen values of A are $\pm\sqrt{2}$ which implies a saddle. The directions of the 'straight-line' paths from the saddle.

The eigen vector for eigen value $\pm\sqrt{2}$ are respectively, $\begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$. Then all trajectories decay exponentially along the line $y = (1+\sqrt{2})x$ and grow exponentially along the line $y = (1-\sqrt{2})x$ toward the equilibrium point (1,1).

Finally the phase diagram is put together, where the phase paths in the neighbourhoods of the equilibrium points are now known. The process can be assisted by sketching in the direction fields on the lines x = 0, x = 1, etc., also on the curve 1 - xy = 0 on which the phase paths have zero slopes, and the line y = x on which the paths have infinite slopes.

2.8 Summary

In this unit we have discussed the planar systems and how to classify the equilibrium points of planar system. Some general ideas along with examples are provided for solving linear two dimensional automnomous system. Also we have coined the idea of phase portrait and different cases for drawing a phase portrait are given. Trace determinant method is very useful to classify the equilibrium points for two dimention systems. Basic idea, method and examples for linearization are given.

2.9 Keywords

Planer system, Trace-determinant plane, Stable node, Unstable node, Saddle, Stable spiral, Unstable spiral, Centre, Linearization.

2.10 Further Reading

1. Layek, G. C. (2015). An Introduction to Dynamical Systems and Choas Springer.
2. Perko, L. (2013). Differential equations and dynamical systems (Vol. 7). Springer Science & Business Media.
3. Hirsch, M. W., Devaney, R. L., &; Smale, S. (1974). Differential equations, dynamical systems, and linear algebra (Vol. 60). Academic press.
4. Jordan, D. W., & Smith, P. (1999). Nonlinear ordinary differential equations: an introduction to dynamical systems (Vol. 2). Oxford University Press, USA.
5. Strogatz, S. H. (2018). Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering. CRC Press.
6. Kaplan, D., &; Glass, L. (2012). Understanding nonlinear dynamics. Springer Science h Business Media.
7. Wiggins, S. (2003). Introduction to applied nonlinear dynamical systems and chaos (Vol. 2). Springer Science h Business Media.
8. Kaplan, D., &; Glass, L. (2012). Understanding nonlinear dynamics. Springer Science &; Business Media.

2.11 Exercise

1. Find the general solution of the system $\dot{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where be > 0.

2. For the harmonic oscillator system $\ddot{x} + b\dot{x} + kx = 0$, find all values of b and k for which this system has real, distinct eigenvalues. Find the general solution of this system in these cases. Find the solution of the system that satisfies the initial condition (0,1).

3. Consider the system $\dot{x} = 4x - y, \dot{y} = 2x + y$.

a) Write the system as $\dot{X} = AX$. Show that the characteristic polynomial is $\lambda^2 - 5\lambda + 6 = 0$, and find the eigenvalues and eigenvectors of A.

b) Find the general solution of the system.

c) Classify the fixed point at the origin.

d) Solve the system subject to the initial condition $(x_0, y_0) = (3,4)$.

4. Find and classify the equilibrium points of $x' = \frac{1}{8}(x + y)^3 - y$, $y' = 18(x+y)^3 - x$. Verify that lines $y = x$, $y = 2 - x$, $y = -2 - x$, are phase paths. Finally sketch the phase diagram of the system.

5. Sketch phase diagrams for the following linear systems and classify the equilibrium point:

   a) $\dot{x} = x - by, \dot{y} = x - y$;

   b) $\dot{x} = x + y, \dot{y} = x - 2y$;

   c) $\dot{x} = -4x + 2y, \dot{y} = 3x - 2y$;

   d) $\dot{x} = x - x^3, \dot{y} = -y$;

   e) $\dot{x} = x(2 - x - y), \dot{y} = x - y$.

6. Show that the origin is a spiral point of the system $\dot{x} = -y - x\sqrt{x^2 + y^2}, \dot{y} = x - y\sqrt{x^2 + y^2}$ but a centre for its linear approximation.

7. Use linear stability analysis to classify the fixed points of the following systems.

a) $\dot{x} = 1 - e^{-x^2}$,

b) $\dot{x} = x(1 - x)$.

c) $\dot{x} = \text{Sin}y$, $\dot{y} = \text{Cos}x$.

8. Using linear stability analysis, classify the fixed points of the Gompertz model of tumor growth $\dot{N} = -aN\ \text{In}(6N)$.

9. Find the nature and stability of the fixed points of $\dot{x} = -ax + y$, $\dot{y} = -x - ay$ for different values of the parameter a.

Unit 3 ❑ Stability Analysis of a System

Structure

3.0 Objective

This unit is to give idea on stability of the system, different Liapunov methods to analyze the stability. Stability of a dynamical system is a fundamental requirement for its practical value, particularly in most real-world applications. We will give brief description on periodic solutions and limit cycle. Gradient system and Hamiltonian system are also very useful in real life. One of the basic nonlinear system is pendulum. Motion of pendulum is discussed.

3.1 Introduction

The technical term "stability" first appeared in the context of mechanics in 1749 in the work of Euler. The question of stability of floating bodies has been a strong motivation for the theoretical research on stability in the seventeenth and eighteenth century. Although there is no outright definition of stability, ideas on stability have continuously developed during the course of history. Various stability concepts exist in modern literature, but we shall confined ourself to Liapunov stability mainly.

It's useful to introduce some language that allows us to discuss the stability of different types of fixed points. This language will be especially useful when we analyze fixed points of nonlinear systems.

The stability of an equilibrium point determines whether or not solutions nearby the equilibrium point remain nearby, get closer, or get further away.

Consider a dynamical system represented by ODE as

$$\dot{X} = F(X, t),\ X \in R^n,\ n \geq 1. \tag{3.1}$$

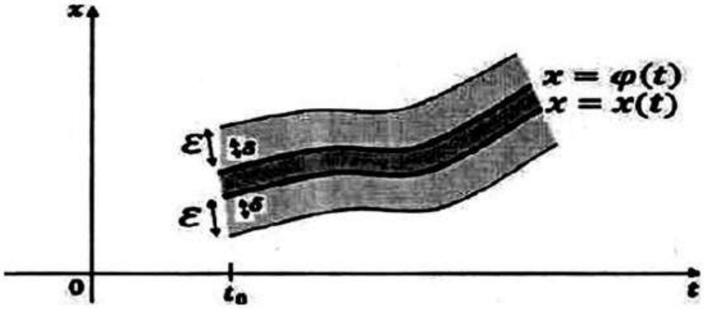
Let X^* is an isolated equilibrium point of the system. We assume that the function $X = \Phi(t)$ is a solution of the equation (3.1), which satisfies the initial condition $[X]_{t=t_0} = \Phi(t_0)$, $t_0 \geq 0$. We assume furthermore, that the function $X = X(t)$ is a solution of the same equation, which satisfies another initial condition $[X]_{t=t_0} = X(t_0)$. It is assumed that the solutions $\Phi(t)$ and $X(t)$ are defined for all $t \geq t_0$.

3.2 Liapunov Stability

An equilibrium point is stable if initial conditions that start near an equilibrium point stay near that equilibrium.point. Mathematically, we say that the solution $X = \Phi(t)$ of the system (3.1) is stable if for all $\in > 0$, there exists an $\delta = \delta(\in) > 0$ such that

$$\|X(t_0) - \Phi(t_0)\| < \delta \Rightarrow \|X(t) - \Phi(t)\| < \in \ \forall\, t > t_0.$$

Note that this definition does not imply that $x(t)$ gets closer to $\phi(t)$ as time increases, but rather just that it stays nearby. Furthermore, the value of δ may depend on $\in$, so that if we wish to stay very close to the equilibrium point, we may have to start very, very close. This type of stability is sometimes called stability "in the sense of Liapunov". In other word, the equilibrium point is called Liapunov stable.



The trivial solution x = 0 of the system $\frac{dx}{dt} = 0$ is Liapunov stable.

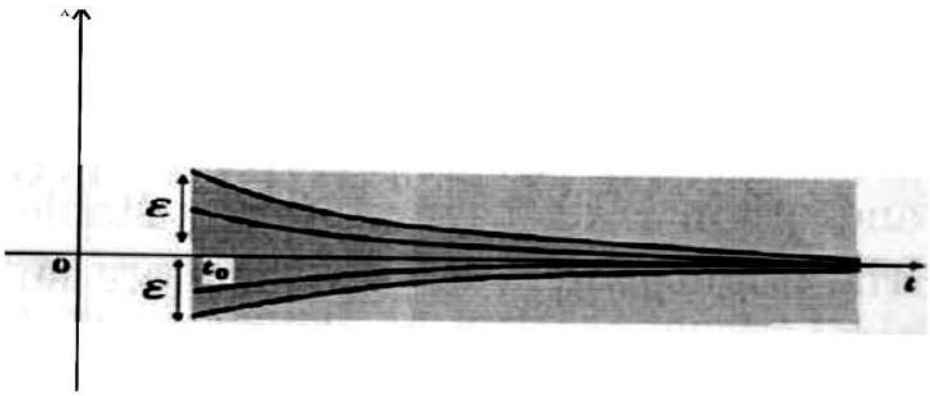
3.3 Asymptotic Stability

The solution $X = \Phi(t)$ of the system (3.1) is said to be asymptotically stable if

(i) the solution $X = \Phi(t)$ is Liapunov stable and

(ii) there exist $\delta > 0$ such that for any solution $X = X(t)$ of (3.1), which satisfies the condition $\|X(t_0) - \Phi(t_0)\| < \delta$, we have $\lim_{t\to\infty} \| X(t) - \Phi(t) \| = 0$.

The trivial solution x = 0 of the equation $\frac{dx}{dt} = 0$ is Liapunov stable but not asymptotically stable. The trivial solution x = 0 of the equation $\frac{dx}{dt} = -a^2x$ (a constant), is asymptotically stable.



Note: The stability of a nontrivial solution of a differential equation does not imply that the solution is bounded. Also the boundedness of solution of a differential equation does not imply that the solutions are stable. The concepts of boundedness and stability of solutions are mutually independent. All solutions of $\frac{dx}{dt} = \sin 2x$ are bounded but $x(t) = 0$ solution of this differential equation is unstable. The solution $x(t) = t$ is a stable solution of the differential equation $\frac{dx}{dt} = 1$ but it is not bounded.

3.4 Liapunov Function

Even for systems that have nothing to do with mechanics, it is occasionally possible to construct an energy-like function that decreases along trajectories. Such a function is called a Liapunov function.

Let V(x) be a continuously differentiable function. Let $\dot{V} = \frac{\partial V}{\partial x}\dot{x}$. Then V is called Liapunov function if V is positive definite, i.e $V(x) > 0$ for all $x \neq x^*$, and $V(x^*) = 0$.

THEOREM 3.4.1 Let V be a real valued function on some open neighbourhood G of equilibrium point and $V(x^*) = 0$. Let V satisfies

1. $V(x) > 0$ and $\dot{V}(x)$ $\forall$ $x \neq x^*$, then x^* is asymptotically stable: for all initial conditions, $x(t) \to x^*$ as $t \to \infty$.

2. $V(x) > 0$ and $\dot{V} \leq 0$ $\forall$ $x \neq x^*$, then x^* is stable but not asymptotically.

Liapunov functions are not unique and hence we can use many different methods to find one. Indeed, one of the main difficulties in using Liapunov functions is finding them. The existence of Liapunov function in a neighbourhood of an equilibrium point but non-existence in the whole space implies the local stability of the equilibrium point.

Note: Liapunov function in certain sense is a generalized distance from the origin. Moreover $V(0) = 0$ is required. Otherwise choosing $V(x) = 1/(1 + |x|)$ we can prove that $x^{\cdot}(t) = x$ is locally stable. But actually the system $x^{\cdot}(t) = x$ is unstable at 0.

Ex 3.4.1 By constructing a Liapunov function, show that the system $\dot{x} = -x + 4y$, $\dot{y} = -x - y^3$ has no closed orbits.

Solution: Now, (0,0) is a equilibrium point for the system. Consider $V(x, y) = x^2+ay^2$, where a is a parameter to be chosen later: Then $\dot{V} = 2x\dot{x} + 2ay\dot{y} - 2x(-x + 4y) + 2ay(-x - y^3) = -2x^2 + (8 - 2a)xy - 2ay^4$. If we choose a = 4, the xy term disappears and $\dot{V} = -2x^2 - 8y^4$. By inspection, $V > 0$ and $\dot{V} < 0$ $\forall$ $(x, y) \neq (0, 0)$.

Hence $V = x^2 + 4y^2$ is a Liapunov function and origin is an asymptotically stable equilibrium point i.e all trajectories approach the origin as $t \to \infty$, so there are no closed orbits.

Ex 3.4.2 Using Liapunov function show that the origin is asymptotically stable equilibrium point of the system $\dot{x} = -2y + yz - x^3$, $\dot{y} = x - xz - y^3$, $\dot{z} - xy - z^3$.

Solution: Let, $V(x, y, x) = x^2 + 2y^2 + z^2$ as Liapunov function for the given system. Now, $\dot{V} = 2x\dot{x} + Ay\dot{y} + 2z\dot{z} = 2x(-2y + yz - x^3) + 4y(x - xz - y^3) + 2z(xy - z^3) = -(2x^4 + 4y^4 + 2z^4)$. Thus, $V(0,0,0) = 0$ and $V > 0$, $\dot{V} < 0$ $\forall(x,y,z) \neq (0,0,0)$.

Hence the equilibrium point origin is asymptotically stable.

3.5 Local and Global Stability

Global stability analysis of an equilibrium point can be done by Liapunov function (without solving differential equation). Origin is a globally asymptotically stable equilibrium point if there exist a Liapunov function for the system in R^n. However, the local stability are analysed based on linearization of the nonlinear system mostly.

3.6 Periodic Solution

A solution of (3.1) through the point x_0 is said to be periodic of period T if there exists $T > 0$ such that $x(t,x_0) = x(t + T, x_0)$ for all $t \in R$. Any periodic orbit in the phase space is a smooth closed curve. Let us consider a nonlinear autonomous system,

$$\dot{x} = F(x,y)$$

$$\dot{y} = G(x,y). \tag{3.2}$$

The function F(x,y) and G(x,y) are continuous and have continuous first order partial derivatives throughout the phase plane. We are interested to find the global properties of paths. Global properties of paths are those which describe their behaviour over large region of the phase plane. The main focus of the global theory is that of finding whether the system (3.2) has closed paths.This is the close connection with the periodic solution of the system (3.2).

3.7 Limit Cycle

A limit cycle is an isolated closed trajectory in the phase space. Isolated means that neighboring trajectories are not closed; they are spiral either toward or away from the limit cycle.

DEFINITION 1 A cycle of a continuous-time dynamical system, in a neighborhood of which there are no other cycles, is called a limit cycle. It is a cycle in a limiting sense.

If all neighboring trajectories approach the limit cycle, we say the limit cycle is stable or attracting. Otherwise, the limit cycle is unstable, or in exceptional cases, half-stable. Limit cycles are inherently nonlinear phenomena; they can't occur in linear systems and its phase space should be at least two dimensional. Of course, a linear system $\dot{x} = Ax$ can have closed orbits, but they won't be isolated; if x(t) is a periodic solution, then so is cx(t) for any constant $c \neq 0$. Hence x(t) is surrounded by a one-parameter family of closed orbits.

3.8 Attractors

An invariant set in the phase space are those sets which remain invariant under time evolution. A closed invariant set $A \subset E$ is called an attracting set of $\dot{x} = f(x)$, where $f \in C^1(E)$ where E is an open subset of R^n, if there is some neighborhood U of A such that $\forall\, x \in U$, $\phi_t(x) \in U$ for all $t \geq 0$ and $\phi_t(x) \to A$ as $t \to \infty$. An attractor of the above system is an attracting set which contains a dense orbit. Attractors can be of different types, viz., point attractor, periodic attractor, strange attractor, etc.

Most common example of an attractor is a free damped pendulum whose phase space trajectory is a spiral converging to the origin. Fixed point and limit cycle are examples of attractor and they are predictable. There is another kind of attractor which is called strange attractor. Strange attractors exhibit unpredictable and unusual motions and hence also called chaotic attractor. The example of strange attractor is the famous Lorenz attractor.

Beside above three attractors there is another type of attractors called 'Torus'.

3.9 Bendixson's Criterion

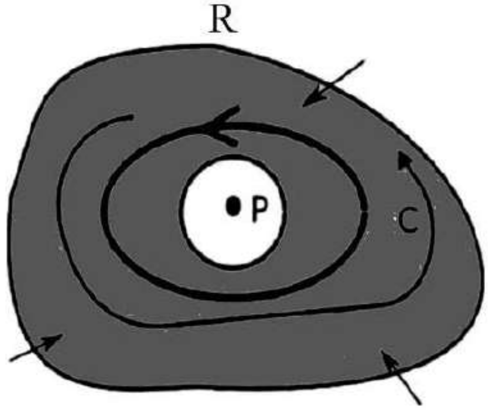
If F_x and G_y of equation (3.2) are continuous in a region R which is simply-connected (i.e., without holes), and $F_x + G_y \neq 0$ at any point of R, then system (3.2) has no closed trajectories inside R.

3.10 Poincare-Bendixson Theorem

The Poincare-Bendixson theorem gives us a complete determination of the asymptotic behaviour of a large class of flows on the plane, cylinder, and two-sphere.

Let R is a closed, bounded subset of the phase plane; $\dot{x} = f(x)$ is a continuously differentiable vector field on an open set containing R; R does not contain any equilibrium points; and there exists a trajectory C that is confined in R, which means the starts in R it stays in R for all future time. Then either C is a closed orbit, or if a trajectory spirals towards a closed orbit as $t \to \infty$. In both the cases, R contains a closed orbit.

If we start on one of the boundary curves, the solution will enter in, since the velocity vector points into the interior of R. As time goes on, the solution can never leave R, since as it approaches a boundary curve, trying to escape from R, the velocity vectors are always pointing inwards, forcing it to stay inside R. Since the solution can never leave R the only thing it can do as $t \to \infty$ is either approach a critical point P but there are none, by hypothesis- or spiral in towards a closed trajectory. Thus there is a closed trajectory inside R.

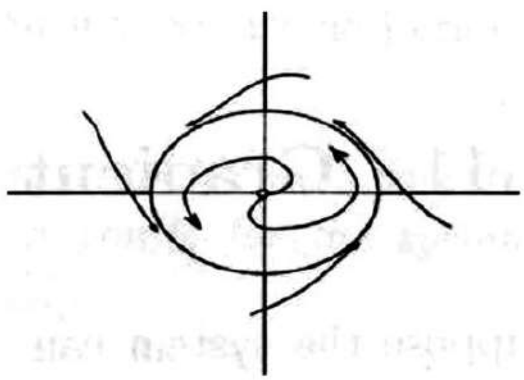


Ex 3.10.1 Show that the system given by $\dot{r} = r(1 - r^2) + \mu r\cos\theta$, $\dot{\theta} = 1$. has a stable limit cycle for $\mu = 0$ at $r = 1$. Also show that a closed orbit still exists for $\mu > 0$, as long as μ is sufficiently small.

Solution: For $\mu = 0$, the radial and angular dynamics are uncoupled and so can be analyzed separately. Treating $\dot{r} = r(1 - r^2)$ as a vector field on the line, we see that $r^* = 0$ is an unstable fixed point and $r^* = 1$ is stable fixed point.

Hence, back in the phase plane, all trajectories (except $r^* = 0$)) approach the unit circle $r^* = 1$ monotonically.

Since the motion in the θ-direction is simply rotation at constant angular velocity, we see that all trajectories spiral asymptotically toward a limit cycle at $r = 1$. Since $\theta > 0$. Therefore flow direction will be anticlockwise.



If we plot the solutions as functions of t, we find that the solution settles down to a sinusoidal oscillation of constant amplitude, corresponding to the limit cycle solution $x(t) = \cos(t + \theta_0)$.

For $\mu > 0$, we seek two concentric circles with radii r_{min} and r_{max} such that $\dot{r} < 0$ on the outer circle and $\dot{r} > 0$ on the inner circle. Then the annulus $0 < r_{min} \le r \le r_{max}$ will be our desired trapping region. Note that there are no fixed points in the annulus since $\dot{\theta} > 0$; hence if r_{min} and r_{max} can be found, the Poincare'Bendixson theorem will imply the existence of a closed orbit.

To find the least value of r, we require $\dot{r} = r(1 - r^2) + \mu r \cos\theta > 0$ for all θ. Since $\cos\theta \geq -1$, a sufficient condition for r_{min} is $1 - r^2 - \mu > 0$. Hence, any $r_{min} < \sqrt{1-\mu}$ will work as long as $\mu < 1$.

We choose $r_{min} = 0.999\sqrt{1-\mu}$. By a similar argument, the flow is inward on the outer circle if $r_{max} = 1.001\sqrt{1+\mu}$. Therefore a closed orbit exists for all $\mu < 1$, and it lies somewhere in the annulus $0.999\sqrt{1-\mu} < r < 1.001\sqrt{1+\mu}$.

Ex 3.10.2 Consider the glycolytic oscillator, $\dot{x} = -x + ay + x^2y$, $\dot{y} = b - ay - x^2y$. Prove that a closed orbit exists if a(> 0) and b(> 0) satisfy an appropriate condition, to be determined.

Solution: The fixed points are obtained from, $\dot{x} = 0$ and $\dot{y} = 0$. Simple calculation gives, $(x^*, y^*) = \left(b, \frac{b}{a+x^2}\right)$. The Jacobian matrix is $A = \begin{pmatrix} -1+2xy & a+x^2 \\ -2xy & -a-x^2 \end{pmatrix}$. The Jacobian has determinant $\Delta = a + b^2$ and trace $\tau = -\frac{b^4 + (2a-1)b^2 + a + a^2}{a+b^2}$.

Hence, the fixed point is unstable for $\tau > 0$, and stable for $\tau < 0$. The dividing line $\tau = 0$ occurs when $b^2 = \frac{1}{2}\left(1 - 2a \pm \sqrt{1-8a}\right)$.

For parameters in the region corresponding to $\tau > 0$, we are guaranteed that the system has a closed orbit-numerical integration shows that it is actually a stable limit cycle. Limit cycle oscillation of the system for a = 0.08, b = 0.6 is shown in Figure 3.1.

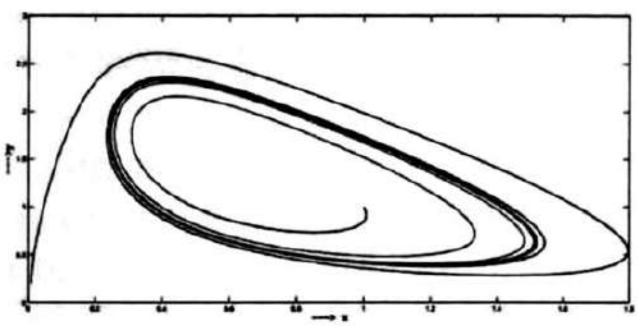


Figure 3.1: Limit cycle solution of glycolytic oscillator with initial condition (a) (1,1) with red line (b) (0.01,0.2) with blue line are shown.

3.11 Gradient Systems

Suppose the system can be written in the form $\dot{x} = -\Delta V$, for some continuously differentiable, single-valued scalar function V(x). Such a system is called a gradient system with potential function V.

Note: For a gradient system, the linearized system at any equilibrium point has only real eigenvalues.

THEOREM 3.11.1 Closed orbits are impossible in gradient systems.

Solution: Suppose there was a closed orbit. We obtain a contradiction by considering the change in V after one circuit. On the one hand, $\Delta V = 0$, since V is single-valued.

But on the other hand,

$$\Delta V = \int_0^T \frac{dv}{dt} dt$$

$$= \int_0^T (\nabla V \dot{x}) dt$$

$$= -\int_0^T \| \dot{x} \|^2 dt < 0 \qquad (3.3)$$

(unless $\dot{x} = 0$, in which case the trajectory .is a fixed point, not a closed orbit). This is a contradiction which shows that closed orbits can't exist in gradient systems.

Ex 3.11.1 Show that the nonlinearly damped oscillator $x^{\cdot\cdot} + (x^{\cdot})^3 + x = 0$ has no periodic solutions.

Solution: Suppose that there were a periodic solution x(t) of period T. Consider the energy function $E(x, \dot{x}) = \frac{1}{2}(r^2 + x^2)$. After one cycle, x and $\dot{x}$ return to their starting values, and therefore $\Delta E = 0$ around any closed orbit.

On the other hand, $\Delta E = \int_0^T$ Edt. If we can show this integral is nonzero, we've reached a contradiction. Note that $\dot{E} = \dot{x}(x + \ddot{x}) = \dot{x}(-\dot{x}^3) = -\dot{x}^4 \leq 0$. Therefore $\Delta E = -\int_0^T \dot{x}^4 dt \leq 0$, with equality only if $x = 0$. But $x = 0$ would mean the trajectory is a fixed point, contrary to the original assumption that it's a closed orbit. Thus ΔE is strictly negative, which contradicts $\Delta E = 0$. Hence there are no periodic solutions.

Ex 3.11.2 Consider a particle of mass m = 1 moving in a double-well potential $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$. Find and classify all the equilibrium points for the system. Then plot the phase portrait and interpret the results physically.

Solution: The force is $-\frac{dV}{dx} = x - x^3$, so the equation of motion is $\ddot{x} = x - x^3$. This can be rewritten as the vector field

$$\dot{x} = y$$
$$\dot{y} = x - x^3$$

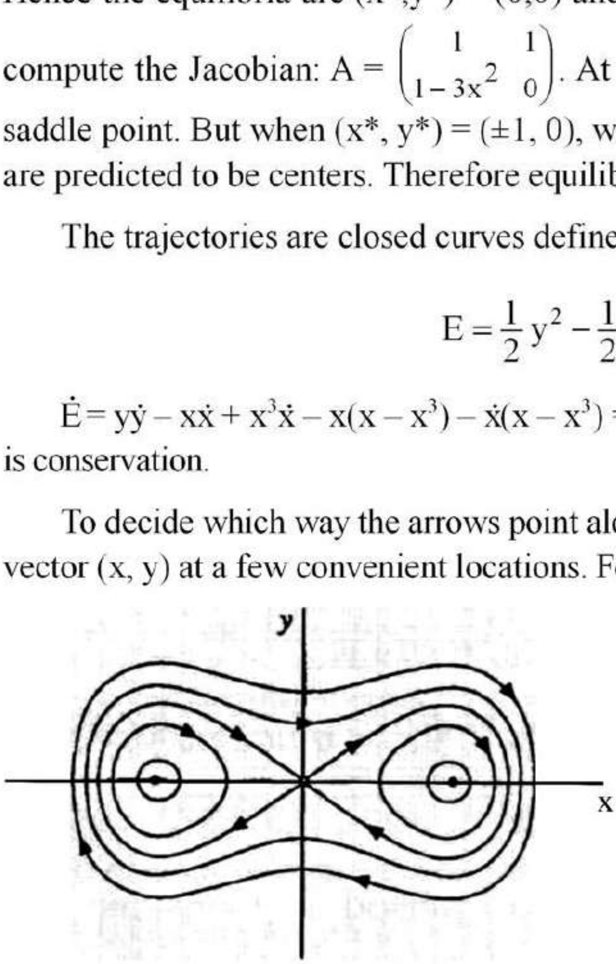
where y represents the particle's velocity. Equilibrium points occur where $(\dot{x},\dot{y}) = (0,0)$. Hence the equilibria are $(x^*,y^*) = (0,0)$ and $(\pm1,0)$. To classify these fixed points we compute the Jacobian: $A = \begin{pmatrix} 1 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$. At (0, 0), we have $\Delta = -1$, so the origin is a saddle point. But when $(x^*, y^*) = (\pm1, 0)$, we find $\tau = 0$, $\Delta = 2$; hence these equilibria are predicted to be centers. Therefore equilibrium points are saddles and centers only.

The trajectories are closed curves defined by the contours of constant energy

$$E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4.$$

$\dot{E} = y\dot{y} - x\dot{x} + x^3\dot{x} - x(x - x^3) - \dot{x}(x - x^3) = 0 \Rightarrow E = \text{constant}$. Therefore, the system is conservation.

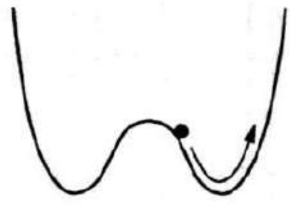
To decide which way the arrows point along the trajectories, we simply compute the vector (x, y) at a few convenient locations. For example, x > 0 and y = 0 on the positive y-axis, so the motion is to the right. The orientation of neighboring trajectories follows by continuity.



As expected, the system has a saddle point at (0, 0) and centers at (1,0) and (-1,0). Each of the neutrally stable centers is surrounded by a family x of small closed orbits. There are also large closed orbits that encircle all three fixed points. Thus solutions of the system are typically periodic, except for the equilibrium solutions and two very special trajectories: these are the trajectories that appear to start and end at the origin. More precisely, these trajectories approach the origin as $t \to \pm\infty$. They are common in conservative systems, but are rare otherwise.

Finally, let's connect the phase portrait to the motion of an undamped particle in a double-well potential.

The neutrally stable equilibria correspond to the particle at rest at the bottom of one of the wells, and the small closed orbits represent small oscillations about these equilibria. The large orbits represent more energetic oscillations that repeatedly take the particle back and forth over the hump.



3.12 Hamiltonian System

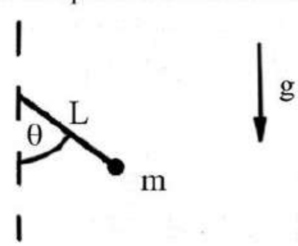
Hamiltonian systems are fundamental to classical mechanics; they provide an equivalent but more geometric version of Newton's laws. They are also central to celestial mechanics and plasma physics, where dissipation can sometimes be neglected on the time scales of interest. The theory of Hamiltonian systems is deep and beautiful, but perhaps too specialized and subtle for a first course on nonlinear dynamics. Hamiltonian system is basic concept for quantum mechanics also.

Let H(p, q) be a smooth, real-valued function of two variables. The variable q is the "generalized coordinate" and p is the "conjugate momentum". Then a system of the form $\dot{q} = \frac{\partial H}{\partial p}$ $\dot{p} = -\frac{\partial H}{\partial q}$ is called a Hamiltonian system and the function H is called the Hamiltonian.

3.13 Motion of Pendulum

Consider a pendulum consisting of a light rod of length L to which is attached a ball of mass m. The other end of the rod is attached to a wall at a point so that the ball of the pendulum moves on a circle centered at this point. The position of the mass at time t is completely described by the angle θ(t) counterclockwise direction. Thus the position of the mass at time t is given by (sinθ(t), – cosθ(t)).

The speed of the mass is the length of the velocity vector, which is $L\frac{d\theta}{dt}$, and the acceleration is $L\frac{d^2\theta}{dt^2}$. We assume that the only two forces acting on the pendulum are the force of gravity and a force due to friction. The gravitational force is a constant force equal to mg acting in the downward direction; the component of this force tangent to the circle of motion is given –mg sinθ. We take the force due to friction to be proportional to



velocity and so this force is given by $-bL\frac{d\theta}{dt}$ for some constant $b > 0$. When there is no force due to friction ($b = 0$), we have an ideal pendulum.

Newton's law then gives the second-order differential equation for the pendulum

$$mL\frac{d^2\theta}{dt^2} = -bL\frac{d\theta}{dt} - mg\sin\theta.$$

For simplicity, we assume that units have been chosen so that m = 1. Rewriting this equation with no friction as a system,

$$\frac{d^2\theta}{dt^2} = -\omega^2 \sin\theta, \tag{3.4}$$

where $\omega = \sqrt{\frac{g}{L}} > 0$. The corresponding system in the phase plane is

$$\begin{aligned}\dot{\theta} &= v\\ \dot{v} &= -\omega^2\sin\theta,\end{aligned} \tag{3.5}$$

v is the (dimensionless) angular velocity.

The fixed points are $(\theta^*, v^*) = (k\pi, 0)$, where k is any integer. There's no physical difference between angles that differ by 2π, so we'll concentrate on the two fixed points (0,0) and $(\pi, 0)$. At (0,0), the Jacobian is $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$. The eigenvalues here are pure imaginary. So the origin is a linear center.

Multiplying (3.5) by $\dot{\theta}$ and integrating yields $\dot{\theta}(\ddot{\theta} + \omega^2 \sin\theta) = 0 \Rightarrow \frac{1}{2}\dot{\theta}^2 \omega^2 \cos\theta =$ constant.

The energy function $E(\theta, v) = \frac{1}{2}v^2 - \omega^2\cos\theta$ has a local minimum at (0,0), since $E \approx \frac{1}{2}\left(v^2 + w^2\theta^2\right) - w^2$ for small (θ, v). $\dot{E} \equiv 0$. E is constant along all solutions of the system. This equation expresses conservation of energy during any particular motion. This equation has the form E = kinetic energy + potential energy at a point, and a particular value of E corresponds to a particular free motion. Hence we may simply plot the level curves of E to see where the solution curves reside. The phase portrait are shown in Figure 3.2.

Now consider the fixed point at $(\pi, 0)$. The Jacobian is $A = \begin{pmatrix} 0 & 1 \\ w^2 & 0 \end{pmatrix}$ The characteristic equation of A is $\lambda^2 - w^2 = 0$. Therefore $\lambda_1 = -w$, $\lambda_2 = w$; the fixed point is a saddle. The corresponding eigenvectors are $v_1 = (-1,w)$ and $v_2 = (1,w)$.

Now consider the family of closed curves immediately surrounding the origin in Figure 3.2 These indicate periodic motions, in which the pendulum swings to and fro about the vertical. The amplitude of the swing is the maximum value of θ encountered on the curve. For small enough amplitudes, the curves represent the usual 'small amplitude' solutions of the pendulum equation in which equation (3.5) is simplified by writing $\sin\theta \equiv \theta$. Then (3.5) is approximated by $\ddot{\theta} + w^2\theta = 0$, having solutions $\theta(t) = A \cos wt + B \sin wt$, with corresponding phase paths $\theta^2 + \frac{v^2}{w^2} = \text{constant}$.

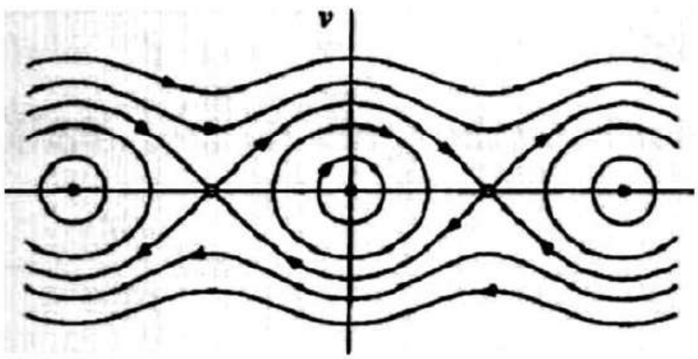


Figure 3.2 :

Ex 3.13.1 Consider the damped linear pendulum given by the equation $\ddot{x} + \dot{x} + x = 0$. Analyze the system.

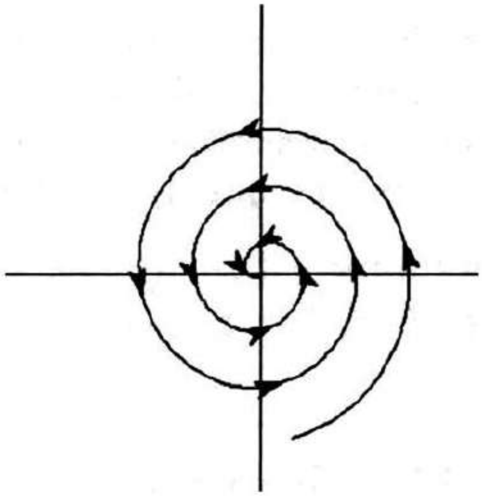
Solution: The equivalent equation is written as

$$\dot{x} = y,$$

$$\dot{y} = -x-y.$$

Origin is equilibrium point of the above system. The Jacobian matrix is given by $A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$.

The characteristic equation is given by $\lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The eigen values are complex with negative real parts, then origin is a stable node, i.e all orbits are spirals converging to the origin. Hence, the origin is an asymptotically stable equilibrium pointed (see the figure below).



Let construct an energy like function as E(x,y) = 0 if (x,y) = 0 and E(x,y) > 0 for (x,y) ≠ 0.

$$\frac{dE}{dt} = \frac{\partial E dx}{\partial x dt} + \frac{\partial E dy}{\partial y dt}$$

$$= x\dot{x} + y\dot{y}$$

$$= x.y + y(-x-y) = -y^2 (< 0).$$

In this system energy function decreases with time and tends to zero with enhancement of time, i.e the system is dissipative.

3.14 Index Theory

The index of a closed curve C is an integer that measures the winding of the vector field on C. The index also provides information about any fixed points that might happen to lie inside a closed curve, as we'll see. This idea may remind you of a concept in electrostatics. In that subject, one often introduces a hypothetical closed surface (a "Gaussian surface") to probe a configuration of electric charges.

Properties of Index

Now we list some of the most important properties of the index.

1. Suppose that C can be continuously deformed into C′ without passing through a fixed point. Then $I_C = I_{C'}$.
2. If C doesn't enclose any fixed points, then $I_C = 0$.
3. If we reverse all the arrows in the vector field by changing $t \rightarrow -t$, the index is unchanged.
4. Suppose that the closed curve C is actually a trajectory for the system, i.e., C is a closed orbit. Then $I_C = +1$.

Note: Notice that the vector field is everywhere tangent to C, because C is a trajectory. Hence, as x winds around C once, the tangent vector also rotates once in the same sense.

Index of a point

Suppose x* is an isolated fixed point. Then the index I of x* is defined as I_C, where C is any closed curve that encloses x* and no other fixed points. By property (1) above, I_C is independent of C and is therefore a property of x* alone. Therefore, we may drop the subscript C and use the notation I for the index of a point.

Ex 3.14.1 Find the index of a stable node, an unstable node, and a saddle point.

Solution: As we traverse C once counter clockwise, the vectors rotate through one full turn in the same sense. Hence $I_C = +1$. The index is also +1 for an unstable node, because the only difference is that all the arrows are reversed; by property (3), this doesn't change the index! (This observation shows that the index is not related to stability). Finally, $I_C = -1$ for a saddle point.

THEOREM 3.14.1 If a closed curve C surrounds n isolated fixed points $x_1^*, x_2^*, ... x_n^*$, then $I_C = I_1 + I_2 + ... I_n$ where I_k, is the index of x_k^*, for k = 1, ... n.

3.15 Keywords

Liapunov stability, Asymptotically stability, Liapunov function, Limit cycle, Poincare Bedixson theorem, Gradient system, Hamiltonian system, Motion of pendulum, Index theory.

3.16 Summary

In this unit we have discussed different types of stabilities e.g Liapunov stability, asymptotic stability, local and global stabilities of equilibrium points. The concepts and examples of periodic solution and limit cycles. Poincare-Bendixson theorem is a milestone regarding periodic solution. Gradient and Hamiltonian systems and their application in finding periodic orbits has been discussed and corresponding examples are provide. A double well potential system is a kind of gradient system. Motion of pendulum and index theory has also discussed.

3.17 Further Readings

1. Strogatz, S. H. (2018). Nonlinear Dynamics and Chaos: with applications to physics, biology, chemistry, and engineering. CRC Press.

2. Layek, G. C. (2015). An Introduction to Dynamical Systems and Chaos, Springir.
3. Jordan, D. W., & Smith, P. (1999). Nonlinear ordinary differential equations: an introduction to dynamical systems (Vol. 2). Oxford University Press, USA.
4. Hirsch, M. W., Devaney, R. L., h Smale, S. (1974). Differential equations, dynamical systems, and linear algebra (Vol. 60). Academic press.
5. Mukherjee, N., h Poria, S. (2012). Preliminary concepts of dynamical systems. Int. J. Appl. Math. Res, 1, 751-770.
6. Alligood, K. T., Sauer, T. D., & Yorke, J. A. (1996). Chaos (pp. 105-147). Springer New York.
7. Wiggins, S. (2003). Introduction to applied nonlinear dynamical systems and chaos (Vol. 2). Springer Science h Business Media.
8. Hilborn, R. C. (2000). Chaos and nonlinear dynamics: an introduction for scientists and engineers. Oxford University Press on Demand.

3.18 Exercise

1. Define Liapunove function and Liapunov stability.
2. Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits, by constructing a Liapunov function $V = ax^2 + by^2$ with suitable a, b.
3. Show that $\dot{x} = -x + 2y^3 - 2y^4$, $\dot{y} = -x - y + xy$ has no periodic solutions.
4. Find the equilibrium points and the general equation for the phase paths of $\ddot{x} + \cos x = 0$. Obtain the equation of the phase path joining two adjacent saddles. Sketch the phase diagram.
5. Use the Poincare Bendixson theorem to show that the vector field $\dot{x} = \mu x - y - x(x^2 + y^2)$, $\dot{y} = x + \mu y - y(x^2 + y^2)$, $(x,y) \in R^2$, has a closed orbit for $\mu > 0$. (Hint: transform to polar coordinates.)
6. Find the approximate relation between amplitude and frequency for the periodic $\ddot{x} - \in x\dot{x} + x = 0$.
7. Find a conserved quantity for the system $\ddot{x} = a - e^x$, and sketch the phase portrait for $a < 0$, $a = 0$, and $a > 0$.
8. (Epidemic model revisited) Consider the model $x^{\cdot} = -kxy$, $y^{\cdot} = kxy - ly$ where $k,l > 0$.

a) Find and classify all the fixed points.

b) Sketch the nullclines and the vector field.

c) Find a conserved quantity for the system. (Hint: Form a differential equation for dy/dx. Separate the variables and integrate both sides.)

d) Plot the phase portrait. What happens as $t \to \infty$.

9. (Harmonic oscillator) For a simple harmonic oscillator of mass m, spring constant k, displacement x, and momentum p, the Hamiltonian is $H = \frac{p^2}{2m} + \frac{kx^2}{2}$. Write out Hamilton's equations explicitly. Show that one equation gives the usual definition of momentum and the other is equivalent to F = ma. Verify that H is the total energy.

10. Find and classify the fixed points of $\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$ for all $b > 0$, and plot the phase portraits for the qualitatively different cases.

11. Show that the system $\dot{x} = y + \frac{x}{\sqrt{x^2 + y^2}}\left\{1 - \left(x^2 + y^2\right)\right\}$,

$\dot{y} = -x + \frac{y}{\sqrt{x^2 + y^2}}\left\{1 - \left(x^2 + y^2\right)\right\}$ has a stable limit cycle.

Unit 4 ❑ Bifurcations and Manifolds

Structure

4.0 Objective

In this unit we will first discuss about hyperbolicity, manifolds and bifurcation. Later we will provide two important continuous dynamical system viz., Lorenz system and Duffing oscillator to discuss there concepts.

4.1 Introduction

A characteristic of nonlinear oscillating systems, a subject of considerable recent interest, is their varieties of responses of which they are capable as change in initial conditions or change of parameter phenomenon. Bifurcation is a phenomenon where the sudden quantitative change in behaviour occurs as a parameter passes through a critical value, called a bifurcation point. A system may contain more than one parameter, each with its own bifurcation points, so that it can display extremely complex behaviour. A manifold is a subspace of a phase or solution space on which a characteristic property such as stability can be associated. First we will discuss the hyperbolicity, stable, unstable and centre manifolds.

4.2 Hyperbolicity

A fixed point or equilibrium point of an nth-order system is hyperbolic if all the eigenvalues of the linearization lie off the imaginary axis, i.e., $Re(\lambda_i) \neq 0$ for i = 1 ...n. Hyperbolic fixed points are sturdy; their stability type is unaffected by small nonlinear terms. On the other hand if an equilibrium point is not hyperbolic then it is called non-hyperbolic.

THEOREM 4.2.1 (Hartman-Grobman theorem): The local phase portrait near a hyperbolic fixed, point is "topologically equivalent" to the phase portrait of the linearization; in particular, the stability type of the fixed point is faithfully captured by the linearization. Here topologically equivalent means that there is a homeomorphism (a continuous deformation with a continuous inverse) that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time (the direction of the arrows) is preserved.

In other words, a phase portrait is structurally stable at hyperbolic equlibrium point i.e. topology of the phase portrait cannot be changed by an arbitrary small perturbation to the vector field. However, at non-hyperbolic equilibrium point phase portrait is structurally unstable and qualitatively different phase portrait arrive for an arbitrarily small perturbation to the vector field.

4.3 Higher-Order Systems: Manifolds

Consider a n-th order autonomous nonlinear systems $\dot{X} = F(X)$, $X, F \in R^n$, which has equilibrium point at $X = X^*$. After linearization about equilibrium point the system becomes

$$\dot{\hat{X}} = A\hat{X}, \tag{4.1}$$

where $A = J(X^*) = [J_{ij}(X^*)] = \left[\frac{\partial F_i(X)}{\partial x_j}\right]_{X=X^*}$, (i,j = 1,...n).

The stability and classification of equilibrium points of linear approximations will depend on the eigenvalues of A. If all the eigenvalues of A have negative real part then the linear approximation is asymptotically stable and so may be is the nonlinear system. If at least one eigenvalue has positive real part then the equilibrium point will be unstable.

Technically a manifold is a subspace of dimension $m \leq n$ in R^n usually satisfying continuity and differentiability. Thus the sphere surface $x^2 + y^2 + z^2 = 1$ is a manifold of dimension 2 in R^3, the solid sphere $x^2 + y^2 + z^2 < 1$ is a manifold of dimension 3 in R^3; and the parabola $y = x^2$ is a manifold of dimension 1 in R^2.

4.3.1 Stable, Unstable and Centre Manifold/Subspaces

The subspaces spanned by the eigen vectors of the matrix A which is associated with the linear system (4.1) determined the stable, unstable and centre manifold or subspace. Let the matrix A has k negative eigen values and n - k positive eigen values and these values are distinct. The set of eigen vectors corresponding to negative eigen values form a k dimensional stable subspace and denoted by W^s, and eigen vectors corresponding to positive eigen values form a n – k dimensional unstable subspace and denoted by W^u. For a nonlinear system the stable manifold will occupy a subset of R^n including a neighbourhood of the origin. If the matrix A has purely imaginary eigen values or zero eigen values then there is also a centre subspace denoted by W^c.

Ex 4.3.1 Find the manifolds for the linear system

$$\dot{x} = -x + 3y,$$

$$\dot{y} = -x + y - z,$$

$$\dot{z} = -y - z.$$

Solution: The eigen values of $\begin{bmatrix} -1 & 3 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$ are given by

$$\begin{bmatrix} -1-\lambda & 3 & 0 \\ -1 & 1-\lambda & -1 \\ 0 & -1 & -1-\lambda \end{bmatrix} = 0 \Rightarrow (\lambda + 1)\ (\lambda + 1) = 0 \Rightarrow \lambda = -1, \pm i.$$

Let $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_3 = -1$: the corresponding eigenvectors are

$$V_1 = \begin{bmatrix} -3 \\ -1+i \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} -3 \\ -1-i \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\lambda_3 = -1$ is the only real eigenvalue and it is negative, it follows that there is no unstable manifold, and that, parametrically, the stable manifold is the straight line $x = -u$, $y = 0$, $z = u(u \in R)$. The centre manifold W^c (which must always have even dimension) is the plane $x + 3z = 0$. In algebraic terminology, we say that E^8 is spanned by V_3, and that E^c is spanned by V_1 and V_2, written as $E^8 = \{V_3\}$, $E^u = \{0\}$, $E^c = \{V_1, V_2\}$.

For a nonlinear system having an equilibrium point with linear manifolds E^8, E^u, and E^c, the actual manifolds W^8, W^u, and W^c are locally tangential to E^8, E^u, and E^c. Whilst solutions on W^8 and W^u behave asymptotically as solutions on E^8 and E^u as $t \to \infty$ respectively, the same is not true of W^c. Solutions on W^c can be stable, unstable, or oscillatory.

4.4 Bifurcations

The term bifurcation was introduced by Poincaré and it has been used to describe significant qualitative changes that occur in the trajectories generally of a nonlinear dynamical system, as the parameters of the system are varied. The parameter values at which the bifurcation occur are called bifurcation points. The methods, techniques and results of bifurcation theory are fundamental to an understanding of nonlinear dynamical systems, and the theories can be applied to any area of nonlinear physics. The bifurcation theory in dynamical system allows one understand many real-world phenomena. Bifurcations occur in both continuous systems and discrete systems.

We begin with most fundamental bifurcations of one dimensional continuous dynamical system.

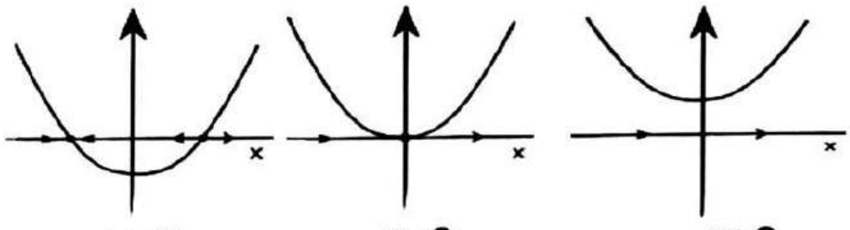
4.4.1 Saddle-Node Bifurcation

The saddle-node bifurcation is the basic mechanism in which the fixed points are created or destroyed as the parameter varies. The prototypical example of saddle-node bifurcation is given by the first order system with parameter r.

$$\dot{x} = f\{x) = r + x^2, \quad r \in R$$

The fixed point of the above one-dimensional system are $x^*_{1,2} = \pm\sqrt{-r}$. When $r < 0$, there are two fixed points. To determine linear stability, we compute $f'(x^*) = 2x^*$.

Thus $x^* = +\sqrt{-r}$ is unstable, since $f'(x^*) > 0$. Similarly, $x^* -\sqrt{-r}$ is stable. When r = 0, the fixed points merge at $x^* = 0$.



And when $r > 0$, there are no fixed points. The bifurcation or the qualitative change in the dynamics is occurred at $r = 0$, since the vector field for $r < 0$ and $r > 0$ are qualitatively different. The stable and unstable fixed points merge at $r = 0$ and j^ig^ppear when $r > 0$. The following Figure 4.1 the parameter r vs x^* is called bifurcation diagram of the system, and $r = 0$ is the bifurcation point.

Ex 4.4.1 Show that the first-order system $\dot{x} = r - x - e^{-x}$ undergoes a saddle-node bifurcation as r is varied, and find the value of r at the bifurcation point.

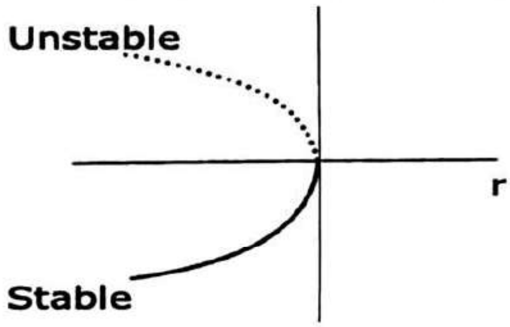


Figure 4.1: Bifurcation diagram

Solution: The fixed points satisfy $f(x) = r - x - e^{-x} = 0$. We can't find the fixed points explicitly as a function of r. We plot $r - x$ and e^{-x} on the same picture. Where the line $r - x$ intersects the curve e^{-x}, we have $r - x = e^{-x}$ and so $f(x) = 0$. Thus, intersections of the line and the curve correspond to fixed points for the system. This picture also allows us to read off the direction of flow on the x-axis: the flow is to the right where the line lies above the curve, since $r - x > e^{-x}$ and therefore $x > 0$. Hence, the fixed point on the right is stable, and the one on the left is unstable.

We start decreasing the parameter r. The line $r - x$ slides down and the fixed points approach each other. At some critical value $r = r_c$, the line becomes tangent to the curve and the fixed points coalesce in a saddle-node bifurcation (Figure 4.2b). For r below this critical value, the line lies below the curve and there are no fixed points (Figure 4.2c).

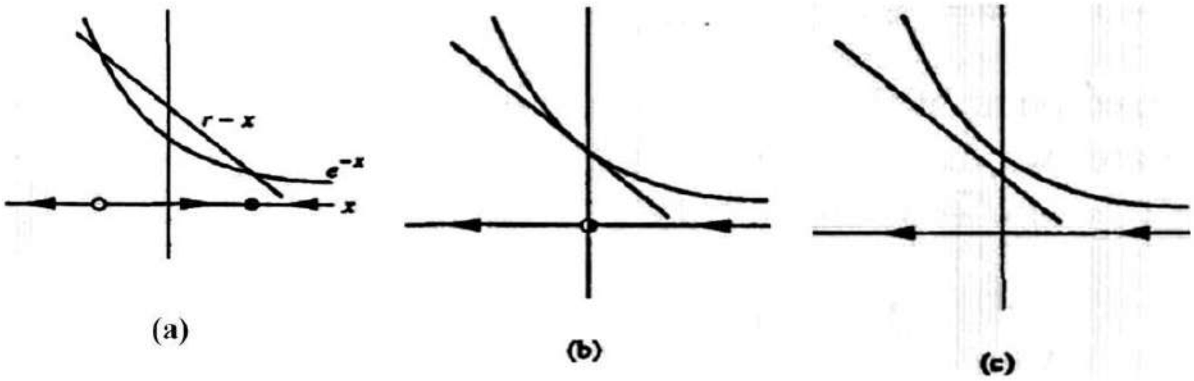


Figure 4.2:

To find the bifurcation point r_c, we impose the condition that the graphs of $r - x$ and e^{-x} intersect tangentially. Thus we demand equality of the functions and their derivatives, $e^{-x} = r - x$ and $\frac{d}{dx}e^{-x} = \frac{d}{dx}(r - x)$ and the second equation implies $-e^{-x} = -1$, so $x = 0$. Then the first equation yields $r = 1$. Hence Saddle-hode the bifurcation point is $r_c = 1$, and the bifurcation occurs at $x = 0$.

4.4.2 Transcritical Bifurcation

There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed. But such fixed point can changes its stability nature as the parameter is varied. The normal form for a transcritical bifurcation is

$$\dot{x} = f(x) = rx - x^2, \qquad r \in R.$$

The fixed points are $x^* = 0$, $x^* = r$ and $f'(x^*) = r - 2x^*$. Note that there is a fixed point at $x^* = 0$ for all values of r.

For $r < 0$, there is an unstable fixed point at $x^* = r$ and a stable fixed point at $x^* = 0$. As r increases, the unstable fixed point approaches the origin, and coalesces with it when $r = 0$. Finally, when $r > 0$, the origin has become unstable, and $x^* = r$ becomes stable. The bifurcation diagram is shown in Figure 4.4.

Ex 4.4.2 Analyze the dynamics of $\dot{x} = r\ln x + x - 1$ near $x = 1$, and show that the system undergoes a transcritical bifurcation at a certain value of r.

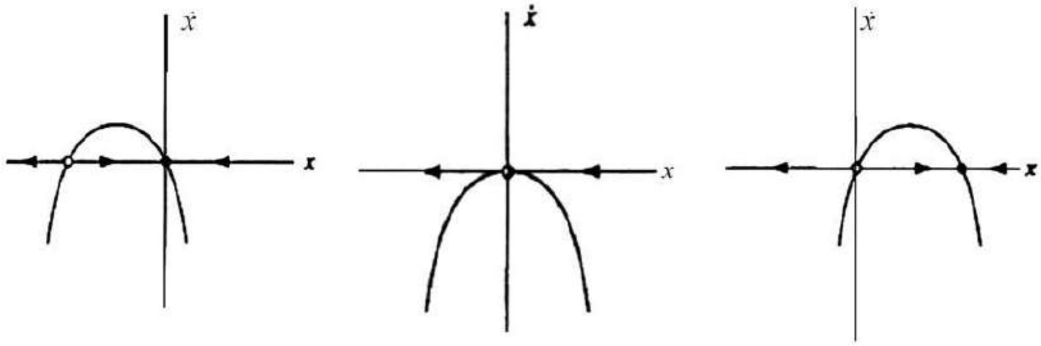


Figure 4.3:

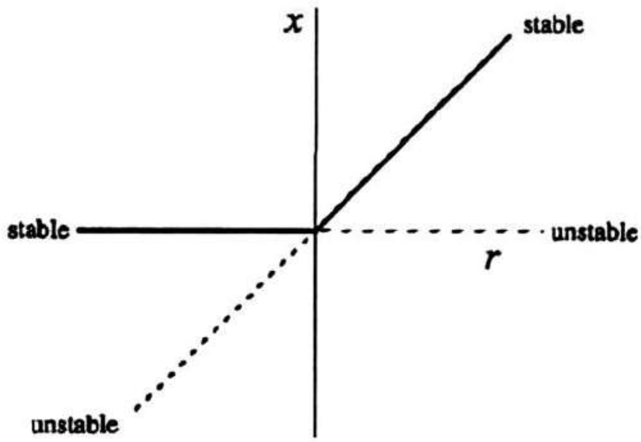


Figure 4.4:

Solution: First note that x = 1 is a fixed point for all values of r. We are interested in the dynamics near this fixed point, we introduce a new variable u = x – 1, where u is small. Then

$$\begin{aligned}\dot{u} &= \dot{x}\\ &= r\ln(u+1)+u\end{aligned}$$

$$= r\left[u - \frac{1}{2}u^2 + O(u^3)\right] + u$$

$$\approx (r+1)u - \frac{1}{2}u^2 + O(u^3)$$

Hence a transcritical bifurcation occurs at $r_c = -1$.

4.4.3 Pitchfork Bifurcation

We turn now to a third kind of bifurcation, the so-called pitchfork bifurcation. This bifurcation is common in many physical system which have symmetry. Many physical system has a special symmetry between left and right. In such cases, fixed points tend to appear and disappear in symmetrical pairs.

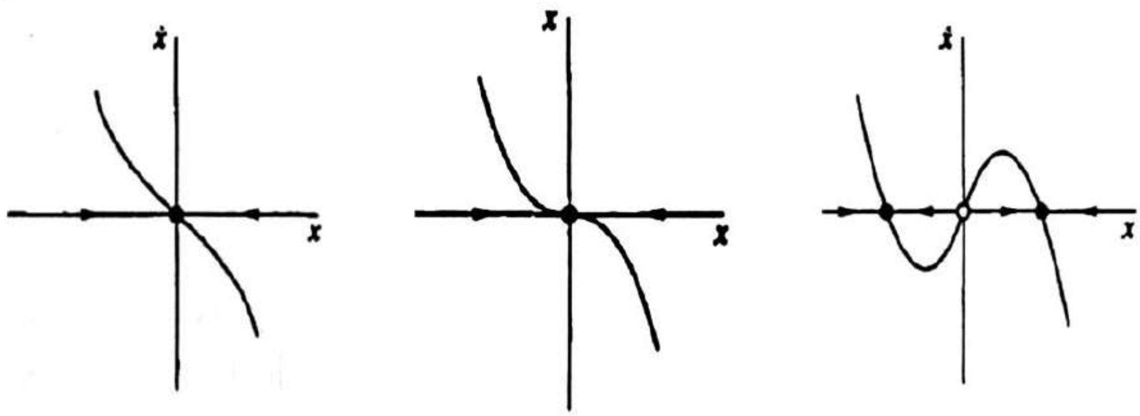
There are two very different types of pitchfork bifurcation. The simpler type is called supercritical, and will be discussed first.

Supercritical Pitchfork Bifurcation

The normal form of pitchfork bifurcation is

$$\dot{x} = f(x) = rx - x^3, \quad r \in R.$$

This equation is invariant under the change of variables $x \to -x$. The fixed points are $x^* = 0, \pm\sqrt{r}$. $f'(x^* = 0) = r$, $f'(x^* = \sqrt{r}) = -2r$, and $f'(x^* = -\sqrt{r}) = 2r$. Figure 4.5



(a) $r<0$ (b) $r = 0$ (c) $r>0$

Figure 4.5:

shows the vector field for different values of r.

When $r < 0$, the origin is the only fixed point, and it is stable.

When $r = 0$, the origin is still stable, but much more weakly so, since the linearization vanishes. Now solutions no longer decay exponentially fast-instead the decay is a much slower algebraic function of time. This lethargic decay is called critical slowing down in the physics literature.

Finally, when $r > 0$, the origin becomes unstable. Two new fixed points appear on either side of the origin, symmetrically located al $x^* = \pm\sqrt{r}$. The bifurcation diagram is shown in Figure 4.6.

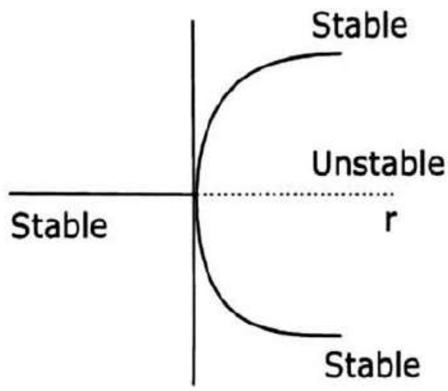


Figure 4.6:

Subcritical Pitchfork Bifurcation

Normal form for subcritical bifurcation

$\dot{x} = f\{x) = rx + x^3, \qquad r \in R.$

Figure 4.7 shows the bifurcation diagram.

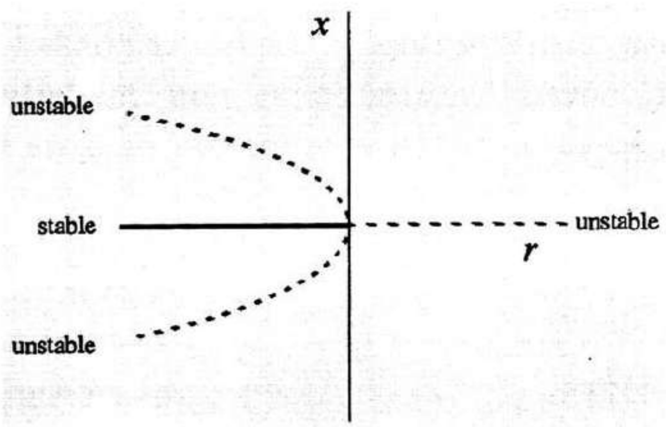


Figure 4.7:

Compared to Figure 4.6, the pitchfork is inverted. The nonzero fixed points $x^* = \pm\sqrt{r}$ are unstable, and exist only below the bifurcation ($r < 0$), which motivates the term "subcritical." More importantly, the origin is stable for $r < 0$ and unstable for $r > 0$, as in the supercritical case, but now the insta-bility for $r > 0$ is not opposed by the cubic term-in fact the cubic term helping the trajectories out to infinity!

4.5 Hopf Bifurcation

There is one more "generic" bifurcation known as Hop-f bifurcation. In an Andronov-Hopf bifurcation (often shortened to Hopf bifurcation), a family of periodic orbits bifurcates from a path of equilibria.

In two and higher dimensional systems saddle-node, transcritical and pitchfork bifurcation are possible. On the other hand in one dimension autonomous system, Hopf bifurcation is impossible since oscillation is impossible in one dimensional autonomous system defined in real line. Therefore for the existence of oscillatory (isolated periodic solutions) solution at-least two dimensional system is necessary (in R^n with $n \geq 2$).

Consider the system

$$\dot{x} = \mu x - y - x(x^2 + y^2)$$
$$\dot{y} = x + \mu y - y(x^2 + y^2) \qquad (4.2)$$

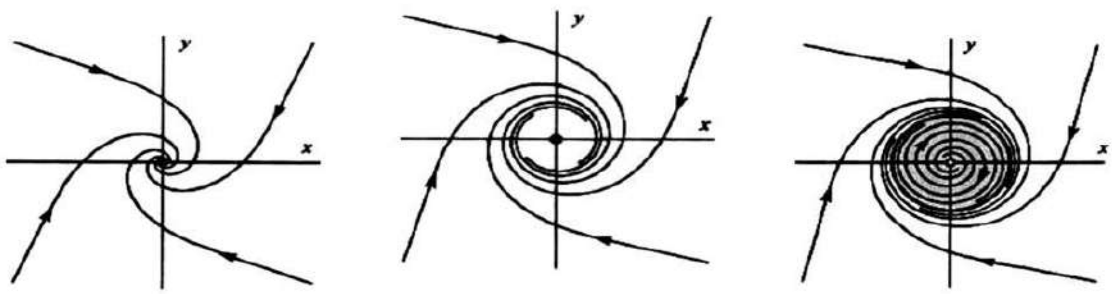
where μ is the bifurcation parameter. There is an equilibrium point at the origin and the linearized system is

$$\dot{x} = \mu x - y$$
$$\dot{y} = x + \mu y.$$

The Jacobian is $\begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$. The eigenvalues are $\mu \pm i$.

In polar coordinates the equations become $\dot{r} = r(\mu - r^2)$, $\dot{\theta} = -1$.

Note that the origin is the only equilibrium point for this system, since $\dot{\theta} \neq 0$. If $\mu \leq 0$ the origin is a sink since $\mu r - r^3 < 0$ for all $r > 0$. Thus all solutions tend to the origin in this case. So the entire diagram consists of a stable spiral. When $\mu > 0$ the equilibrium becomes a source. When $\mu > 0$ we have $\dot{r} = 0$ if $r = \sqrt{\mu}$. So the circle of radius $\sqrt{\mu}$ is a periodic solution with period 2π. We also have $\dot{r} > 0$ if $0 < r < \sqrt{\mu}$, while $\dot{r} < 0$ if $r > \sqrt{\mu}$. Thus, all non-zero solutions spiral toward this circular solution as $t \to \infty$. Then there is an unstable spiral at the origin surrounded by a stable limit cycle which grows out of the origin, the steps in its development are shown in Figure 4.8. This is an example of a Hopf bifurcation which generates a limit cycle. There are three different types of Hopf bifurcation e.g. supercritical, sub-critical and degenarate.



$\mu < 0$ $\mu = 0$ $\mu > 0$

Figure 4.8:

4.6 Lorenz System

Edward Lorenz (1963) had derived the three-dimensional system from a drastically simplified model of convection roll in the atmosphere. The system looks like

$$\dot{x} = \sigma(y\text{-}x),$$
$$\dot{y} = rx - y - xz,$$
$$\dot{z} = xy - bz. \tag{4.3}$$

Here σ, r, b are the positive parameters. The same equations also arise in models of lasers and dynamos. The simple-looking deterministic system has wide range of behaviors depending on parameters. The solutions oscillate irregularly, never exactly repeating but always remaining in a bounded region of phase space. The trajectories in three- dimensional phase space settled into a complicated set, now called strange attractor. Unlike stable fixed points and limit cycles, the strange attractor is not a point or a curve or surface.

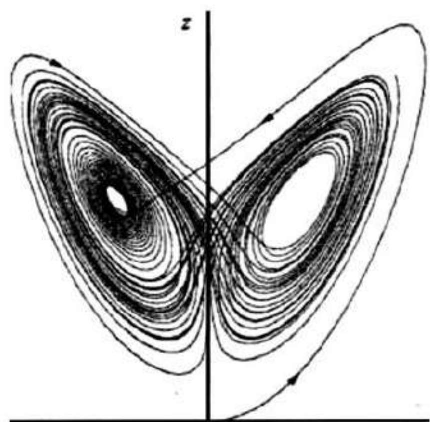


Figure 4.9:

If we replace $(x,y) \rightarrow (-x,-y)$ in (4.3), the equations stay the same. Hence, if $(x(t),y(t), z(t))$ is a solution, so is $(-x(t),-y(t),-z(t))$. In other words, all solutions are either symmetric themselves, or have a symmetric partner.

The Lorenz system is dissipative, which means that the volume in the phase space contract under the flow.

The Lorenz system (4.3) has two types of fixed points. The origin (0,0, 0) is a fixed point for all values of parameters. For $r > 1$, there is also a symmetric pair of foxed points $x^* = y^* = \pm\sqrt{b(r-1)}$, $z^* = r - 1$. Lorenz called them C^+ and C^-. As the parameter $r \to 1$, the two fixed points C^+ and C^- coincide with origin giving a pitchfork bifurcation of the system.

Note: Two different solutions start out very differently, but eventually have more or less the same fate: They both seem to wind around a pair of points, alternating at times which point they encircle. This is the first important fact about the Lorenz system: All non-equilibrium solutions tend eventually to the same complicated set, the so-called Lorenz attractor (Figure 4.9).

Linear Stability of the Origin

The linearization of the system (4.3) at origin is given by

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y, \\ \dot{z} &= -bz. \end{aligned} \tag{4.4}$$

The equation for 2 is decoupled and shows that $z(t) \to 0$ as $t \to \infty$. The other two directions are governed by the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.5}$$

If $r > 1$, the origin is a saddled point because determinant $\Delta = \sigma(r - 1) < 0$. A new type of saddle is created since the system is three-dimensional. Including the decaying 2-direction, the saddle has one outgoing and two incoming directions. If $r < 1$, all directions are incoming and the origin is sink. Specifically, since $\tau^2 - 4\Delta = (\sigma - 1)^2 + 4\sigma r > 0$, the origin is a stable node for $r < 1$.

Global Stability of Origin

To show the global stability of origin we would try to construct a Liapunov function, a smooth, positive definite function that decreases along trajectories.

Let consider a Liapunov function $V(x,y,z) = \dfrac{x^2}{\sigma} + y^2 + z^2$, which is the surfaces of constant V are concentric ellipsoids about the origin.

The idea is to show that if r < 1 and (x, y, 2) $\neq$ (0,0, 0), then $\dot{V} < 0$ along trajectories. This would imply that the trajectory keeps moving to lower V, and hence penetrates smaller and smaller ellipsoids as $t \to \infty$. But V is bounded below by 0, so V(x(i)) ->o 0 and hence $x(t) \to 0$, as desired.

Now calculate:

$$\frac{1}{2}\dot{V} = \frac{x\dot{x}}{\sigma} + y\dot{y} + z\dot{z}$$

$$= (xy - x^2) + (rxy - y^2 - xyz) + (xyz - bz^2)$$

$$= (r + 1)xy - x^2 - y^2 - bz^2$$

$$= -\left[x - \frac{r+1}{2}y\right]^2 - \left[1 - \left(\frac{r+1}{2}\right)^2\right]y^2 - bz^2$$

$$< 0 \text{ if } r < 1 \text{ and } (x,y,z) \neq (0,0,0)$$

For $\dot{V} = 0$ implies (x, y, z) = (0,0, 0). Otherwise $\dot{V} < 0$. Hence the claim is established, and therefore the origin is globally stable for r < 1.

4.7 Duffing Oscillator

The Duffing oscillator has the dynamical equation

$$\dot{x} = y,$$

$$\dot{y} = x - x^3 - \delta y, \tag{4.6}$$

where δ is a positive parameter. It is easy to see that this equation has three fixed points given by (0,0), ($\pm$1, 0). The matrix associated with the linearized vector field is given by $\begin{pmatrix} 0 & 1 \\ 1-3x^2 & -\delta \end{pmatrix}$. Here, $\Delta = 3x^2 - 1$, $\tau = -\delta$, and $\tau^2 - 4\Delta = \delta^2 - 12x^2 + 4$.

For the fixed point (0,0), $\Delta = -1$ (< 0) which implies the eigen values at (0,0) are real distinct and of opposite sign, so (0,0) is a saddle fixed point. Hence it is unstable.

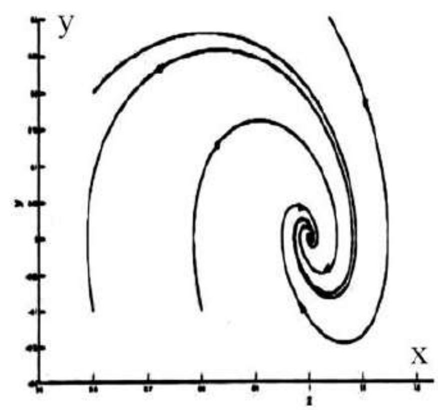


Figure 4.10: Phase diagram of system with $\delta = 0.25$.

For the fixed points ($\pm$1,0) $\Delta = 2(> 0)$, $\tau^2 - 4\Delta = \delta^2 - 8$ and $\tau < 0$. So if $\delta^2 \geq 8$ the fixed points are stable node and if $\delta^2 < 8$ the fixed points are stable spiral (Figure 4.10).

Forced Duffing Oscillator

Forced Duffing Oscillator equation is given by

$$\ddot{x} + \in k\dot{x} - x + x^3 = \in F \cos wt, \ \dot{x} = y,$$

where $0 < \in << 1$. We are now interested in the unstable limit cycle about the origin in the phase plane, and the stable and unstable manifolds associated with points on it. For small $|x|$, x satisfies $\ddot{x} \in k\dot{x} - x = \in F \cos wt$. The periodic solution of this equation is

$$xp = C \cos wt + D \sin wt, \text{ where } C = \frac{-\in(1+w^2)}{(1+w^2)^2 + \in^2 k^2 w^2} \text{ and } D = \frac{\in^2 wk}{(1+w^2)^2 + \in^2 k^2 w^2}$$

which has the fixed point $\left(\frac{-\in}{1+w^2}, \frac{\in^2 kw^2}{(1+w^2)^2}\right)$ to order $\in^3$.

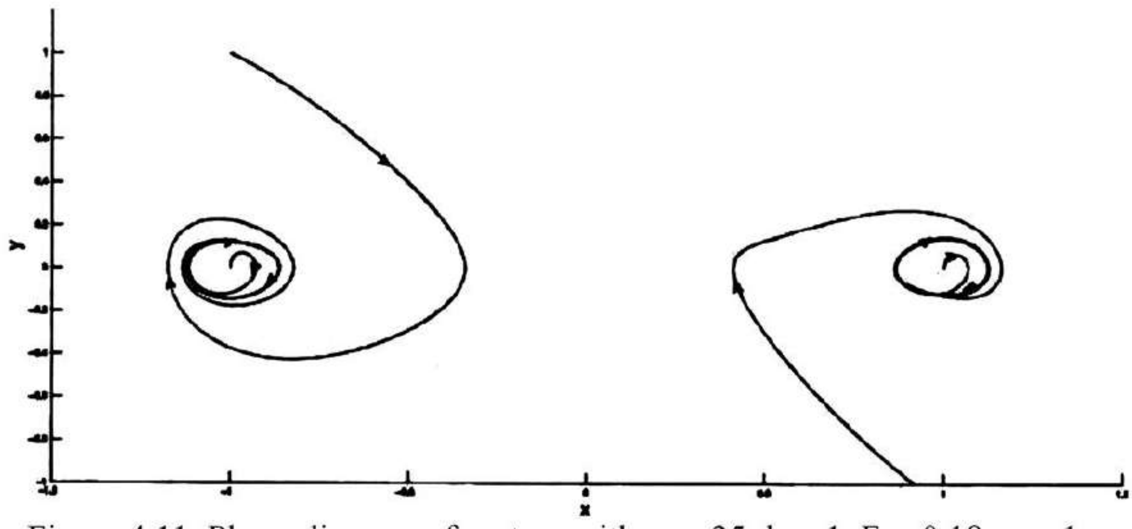


Figure 4.11: Phase diagram of system with $\in = .25$, $k = 1$, $F = 0.18$, $u = 1$.

4.8 Keywords

Hyperbolicity, Stable manifold, Unstable manifold, Centre manifold, Bifurcation, Saddlenode bifurcation, Transcritical bifurcation, Pitchfork bifurcation, Hopf bifurcation, Lorenz system, Duffing oscillator.

4.9 Summary

In this unit we first discuss brief about hyperbolicity of the system. Next we have coined some ideas about stable, unstable and centre manifold of higher order system. Next we come to important concept of bifurcation. Details of saddle node, transcritical and pitchfork bifurcation diagram are given and bifurcation diagram are shown for each case. Also, one of two dimensional bifurcations, Hopf bifurcation is also analysed. A detailed analysis of Lorenz System and Duffing equation is provided.

4.10 Further Readings

1. Layek, G. C. (2015). An Introduction to Dynamical Systems and Chaos, springer.
2. Jordan, D. W., &; Smith, P. (1999). Nonlinear ordinary differential equations: an introduction to dynamical systems (Vol. 2). Oxford University Press, USA.
3. Strogatz, S. H. (2018). Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering. CRC Press.
4. Hirsch, M. W., Devaney, R. L., k Smale, S. (1974). Differential equations, dynamical systems, and linear algebra (Vol. 60). Academic press.
5. Kaplan, D., k Glass, L. (2012). Understanding nonlinear dynamics. Springer Science & Business Media.
6. Mukherjee, N., k Poria, S. (2012). Preliminary concepts of dynamical systems. Int. J. Appl. Math. Res, 1, 751-770.
7. Socolar, J. E. (2006). Nonlinear dynamical systems. In Complex systems science in biomedicine (pp. 115-140). Springer, Boston, MA.
8. Alligood, K. T., Sauer, T. D., k Yorke, J. A. (1996). Chaos (pp. 105-147). Springer New York.

4.11 Excercise

1. For each of the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. And mention the name of bifurcation for each case.

   a) $\dot{x} = rx - \sinh x$

b) $\dot{x} = r + x - \ln(1 + x)$

c) $\dot{x} = x(r - e^x)$

d) $\dot{x} = x + \dfrac{rx}{1+x^2}$.

2. Consider the system $\dot{x} = rx - \sin x$.

a) For the case r = 0, find and classify all the fixed points, and sketch the vector field.

b) Show that when r > 1, there is only one fixed point. What kind of fixed point is it?

c) As r decreases from oo to 0, classify all the bifurcations that occur.

d) For $0 < r << 1$, find an approximate formula for values of r at which bifurcations occur.

e) Now classify all the bifurcations that occur as r decreases 0 to $-\infty$.

f) Plot the bifurcation diagram for r, and indicate the stability of the various branches of fixed points.

3. Discuss all bifurcations of the system $\dot{x} = x^2 + y^2 - 2$, $\dot{y} = y - x^2 + \mu$. Compute phase diagrams for typical parameter values.

4. Let $\dot{x} = \mu x - y + x / (1 + x^2 + y^2)$, $\dot{y} = x - \mu y + y/(1 + x^2 + y^2)$. Show that the equations display a Hopf bifurcation as $\mu > 0$ decreases through $\mu = 1$. Find the radius of the periodic path for $0 < \mu < 1$.

5. Let $\dot{X} = AX$, where $X = [x,y,z]^t$. Find the eigenvalues and eigenvectors of A in each of the following cases. Describe the stable and unstable manifolds of the origin.

a) $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & -1 \end{pmatrix}$ c) $\begin{pmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{pmatrix}$

6. Show that the equilibrium points C^+ and C^- of Lorenz system are sinks provided $1 < r < rH = \sigma\left(\dfrac{\sigma+b+3}{\sigma-b-1}\right)$.

7. a) Show that the Duffing equation $\ddot{x} + x + \in x^3 = 0$ has a nonlinear center at the origin for all $\in > 0$.

b) If $\in < 0$, show that all trajectories near the origin are closed. What about trajectories that are far from the origin?

Unit 5 ❑ Discrete Systems

Structure

5.0 Objective

Our goal in this unit is to begin the study of discrete dynamical systems. While the study of discrete dynamical systems is a topic that could easily fill this, we will restrict attention here primarily to the portion of this theory that helps us understanding chaotic behavior in one dimension. Later we will discuss about chaos, Liapunov exponent, period-doubling bifurcation, logistic map, tent map and horseshoe map.

5.1 Introduction

Let $P \subset R^m$, $m \in N$; $x_n \in P$, $n \in Z$. Then

$$x_{n+1} = G(x_n), \tag{5.1}$$

where $G : P \to P$ is a discrete dynamical system (or discrete-time dynamical system) and $G = (g_1, g_2, \ldots g_m)$. If G is a nonlinear function then it called a nonlinear dynamical system. Logistic map, Henon map are two very famous discrete dynamical system. In some scientific contexts, it is natural to regard time as discrete. Discrete dynamical system has been used in many realistic field as financial marketing, animal populations modelling and digital electronics.

5.2 Fixed Point of a Map

A point p is said to be fixed point of the map $f : X \to X$ if $f(p) = p$, that is, if p is invariant under f. In other way, p is mapped onto itself by f. Orbit of f at x_0 is the set $\{x_0, f(x_0), f^2/x_0), \ldots$

A fixed point p of a map f is said to be attracting (stable) fixea point if there exists $\in > 0$ such that $\forall x \in (p - \in, p + \in)$ so that $\lim_{n\to\infty} f^n(x) = p$. An attracting fixed point is called sink.

Again, a fixed point p of a map f is said to be repelling (unstable) fixed point if for $\in > 0$ there exists some integer M such that $\forall x \in N_\in(p) \in (p - \in, p + \in)$ so that $f^n(x) \notin N\in(p)$, $n > M$. A repelling fixed point is called source.

THEOREM 5.2.1 Let f be a smooth map, i.e $f \in C^1$ and p be a fixed point of f.

(i) If $|f'(p)| < 1$, then p is an attracting (stable) fixed point of f, known as sink;

(ii) if $|f'(p)| > 1$, then p is an repelling (unstable) fixed point of f, known as source.

Proof: (i) We have $|f'(p)| < 1$, there exists m in $0 < m < 1$ such that $|f'(p)| < m < 1$. Since f' is continuous at $x = p$ then for $m > 0$ there exists $\in > 0$ such that

$$\left|\frac{f(x) - f(p)}{x - p}\right| < m \text{ when } |x - p| < \in,$$

$\Rightarrow |f(x) - f(p)| < m|x - p| < 1.$

Now, $|f^2(x) - f^2(p)| = |f(f(x)) - f(f(p))| < m|f(x) - f(p)| < m^2|x - p|$. Similarly, $|f^n(x) - f^n(p)| < m^n|x - p|$. Also we have $0 < m < 1 \Rightarrow m^n \to 0$ as $n \to \infty$. Now, $f^n(p) = p$, then $|f^n(x) - p| \to 0$ as $n \to \infty$ when $|x - p| \in \Rightarrow$ p is a attracting fixed point or sink.

Similarly, case (ii) can be established.

Hyperbolic fixed point: Let f be a map on $R^m, m \geq 1$. Assume that $f(p) = p$. Then the fixed point p is called hyperbolic if none of the eigen values of the Jacobian matrix at p has magnitude 1. Otherwise the fixed point is called non-hyperbolic.

For example, consider the map $f(x) = x^2$. The fixed points are $x^* = 0,1$. Since $|f'(0)| = 0 \neq 1$ and $|f'(1)| = 2 \neq 1$, the fixed points 0,1 are hyperbolic.

5.3 Periodic Points

Let $f : X \to X$ be a map. We call p a periodic point of period k if $f^k(p) = p$ and if k is the smallest such positive integer. The orbit with initial point p is called the periodic orbit k. We often use the abbreviated terms as period k point, or period k orbit or periodic k cycle. If p is a periodic point of period 2 for the map /, then p is a fixed point of the map f^2. However the converse is not true.

The fixed point p_1 with period k is stable if $|(f^k)'(p_1)| < 1$ and is unstable if $|(f^k)'(p_1)| > 1$. The periodic orbit $\{p_1, p_2, \ldots p_k\}$ is stable if $|f'(p_1)f'(p_2)\ldots f'(p_k)| < 1$ and is unstable if $|f'(p_1)f'(p_2)\ldots f'(p_k)| > 1$. Periodic cycle is a collective property, and each points must be distinct.

Ex 5.3.1 Consider the map $f\{x) = x^2 - 1$. Show that the map $f\{x)$ has a periodic orbit of period 2.

Solution: The 2-cycle or period 2 orbit of a map exists if and only if there are two points p and q such that $f(p) = q$ and $f(q) = p$. Equivalently, such a p-must satisfy, $fof(p) = p$ and $fof(q) = q$, i.e., $f^2(p) = p$ and $f^2(q) = q$.

The fixed points of the map are given by $f(x) = (x^2 - 1)$ implies $x = \frac{1 \pm \sqrt{5}}{2}$. Also, $f(0) = -1$, $f(-l) = 0$. $f^2(0) = fof(x) = (x^2 - 1)^2 - 1 = x^4 - 2x^2 = 0$. Similarly, $f^2(-l) = -1$, $f^3(0) = -1$. Hence the map $f = (x^2 - 1)$ has a periodic orbit of period 2, i.e., $\{0, -1\}$ is a periodic orbit of period -2.

5.4 Period-Doubling Bifurcation

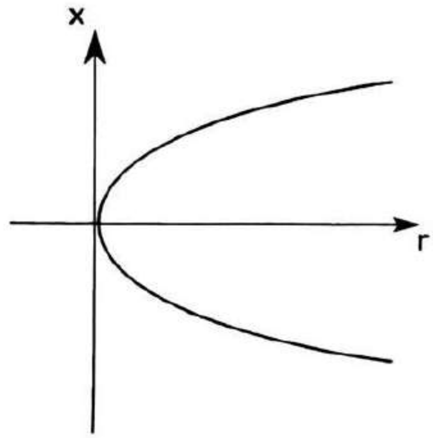
Discrete dynamical systems undergo bifurcations when parameters are varied just as differential equations do.

Flip bifurcation or period-doubling bifurcation is a typical feature of nonlinear maps, rarely observed in continuous systems. The period-doubling bifurcation sequence is as follows : Period $\to$ Period-2 $\to$ Period-2^2 $\to$.......$\to$ Period-infinitum. The normal form of flip bifurcation for a one dimensional discrete map is given by

$$f(x) = -(1 + r)x + x^3, \; r \in R, \; x \in R. \qquad (5.2)$$

The above map (5.2) has fixed points $x^* = 0, \pm\sqrt{2+r}$.

It is easy to verify that (5.2) has a nonhyperbolic fixed point at $x^* = 0$ for $r = 0$ with eigenvalue -1, i.e $f(0) = 0$ and $f'(0) = -1$. The fixed point $x^* = 0$ is unstable for $r \leq -2$, stable for $-2 < r < 0$, unstable for $r > 0$. And $x^{*2} = 2 + r$ is unstable for $r \geq -2$, does not exist for $r < -2$. So, all of three fixed points are unstable for $r > 0$. A way out of this difficulty would be provided if stable periodic orbits bifurcated from $x^* = 0$ for $r = 0$.



Let consider the second iterate

$$f^2(x) = x + r(2 + r)r - 2x^3 + 0(4). \tag{5.3}$$

It is easy to verify that (5.3) has a nonhyperbolic fixed point at $x^* = 0$ for $r = 0$ having an eigenvalue of 1, i.e., $f^2(0) = 0$, $f^{2\prime}(0) = 1$. The second iterate of (5.2) undergoes a pitchfork bifurcation at $x^* = 0$ for $r = 0$. Since the new fixed points of $f^2(x)$ are not fixed points of $f(x)$, they must be period two points of $f(x)$. Hence, $f(x)$ is said to have undergone a period-doubling bifurcation at $x^* = 0$ for $r = 0$.

5.5 Sensitive Dependence on Initial Condition

Sensitive dependence on initial conditions (SDIC) refers to the property that pairs of points which begin as close together as desired, will eventually move apart. Let $f : X \rightarrow X$ be a map. A point $x_0 \in X$ has sensitive dependence on initial conditions if there is a non-zero distanced such that some points arbitrarily near x_0 are eventually mapped at least d units from the corresponding image of x_0. More precisely, there exists $d > 0$ such that any nbd N of x_0 contains a point x such that $|f^k(x) - f^k(x_0)| \geq d$ for some non-negative integer k. Sometimes we call such a point x_0, a sensitive point.

Ordinarily, the closer x is to x_0, the larger k will need to be. The point x will be sensitive if it has neighbours as close as desired that eventually move away the prescribeded distance d for some sufficiently large k. The doubling map $g : s \rightarrow s$ on the unit circle s debined by $g(\theta) = 2\theta$ is an example of a map satisfying SDIC property.

5.6 Liapunov Exponent

The Liapunov exponent of a map is useful to determine wheather a map to be chaotic. Liapunov exponent is a measure of sensitive dependence on initial conditions of a dynamical system. We extend the definition of Liapunov exponent to one-dimensional maps. It gives an average measure at the exponential rate of which nearby orbits move apart.

Let x_0 be some given initial condition. We consider a nearby point $x_0 + \delta_0$ where the initial separation δ_0 is extremely small. Let δ_n be the separation after n iterates. If

$$|\delta_n| \approx |\delta_0| e^{n\lambda}, \tag{5.4}$$

then λ is called the Liapunov exponent. A positive Liapunov exponent is a signature of chaos.

Now,

$$\delta_1 = f(x_0 + \delta_0) - f(x_0)$$
$$\vdots$$
$$\delta_n = f^n(x_0 + \delta_0) - f^n(x_0) \tag{5.5}$$

Taking logarithms of both sides of (5.4) and noting (5.5) we get,

$$\lambda \approx \frac{1}{n} \ln \left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n} \ln \left|\frac{f^n(x_0+\delta_0) - f^n(x_0)}{\delta_0}\right| = \frac{1}{n} \left|\left(f^n\right)'(x_0)\right|,$$

where $\delta_0 \to 0$.

By using chain rule of differentiation we can write $(f^n)'(x_0) = \Pi_{i=0}^{n-1} f'(x_i)$. If this expression has a limit as $n \to \infty$, then we have

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)|, \text{ provided the limit exists.}$$

Since $f^k(x_0) = x_k$, HK $\in Z$.

If $\lambda > 0$ then the system has sensitive dependence on x_0 and if $\lambda < 0$ then the system has no sensitive dependence on x_0, i.e., the map is non-chaotic. The Liapunov exponents for stable periodic and superstable cycles are negative, and so these properties are regular.

5.7 Chaos

Choas is a deterministically unpredictable phenomenon. There could be a motion even for a simple system which is erratic, not simply periodic or quasiperiodic.

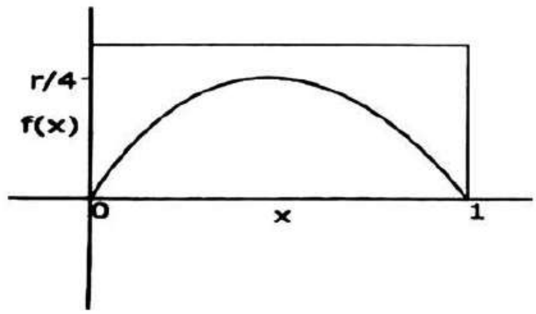
We consider autonomous maps on R^n denoted as $x \to F(x)$. The map $F : I \to I$ is said to be chaotic in an interval I, $I \subset R$, if, (1). F is topologically transitive, given any two subintervals I_1 and I_2 in I there is a point $x_0 \in I_1$ and an $n > 0$ such that $f^n(x_0) \in I_2$, (2), the periodic points of F are dense in I, (3). F has sensitive dependence on initial conditions. This is the Devaney's definition but there are many other definitions of chaos.

5.8 Logistic Map

The logistic map

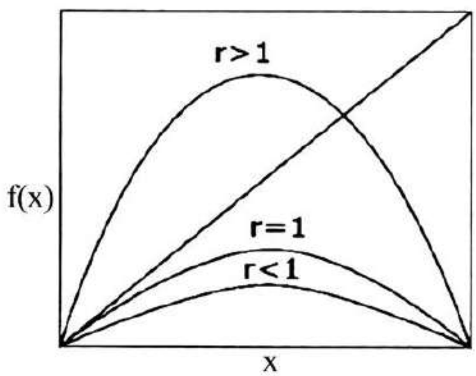
$$x_{n+1} = rx_n(1 - x_n),\ n = 0,1,2,........ \tag{5.6}$$

is a discrete time analogue of the logistic equation for population growth. Here $x_n \geq 0$ is a dimensionless measure of the population in the n-th generation and $r \geq 0$ is the intrinsic growth rate. We restrict the control parameter r to the range $0 \leq r \leq 4$ so that (5.6) maps the interval [0,1] into itself. If $x_n > 1$ for some n, the population becomes negative in the (n + 1) the generation and subsequent iterations diverges towards minus infinity.



The fixed points of the logistic map are obtained by solving the equation

$$rx^*(1 - x^*) = x^*.$$



Therefore, $x^* = 0$, $\left(1-\frac{1}{r}\right)$ are two fixed points of (5.6). The point $\left(1-\frac{1}{r}\right)$ is a fixed point of (5.6) distinct from the fixed point 0 if $r > 1$.

Let x^* be a fixed point of (5.6). To determine the stability of x^*, we calculate derivative of $f(x) = rx(1 - x)$. If $|f'(x^*)| < 1$ then the fixed point is linearly stable. Conversely, if $|f'(x^*)| > 1$, the f(x) fixed point is linearly unstable. The fixed point $x^* = 0$ is locally stable if $r < 1$ and unstable for $r > 1$. At the other fixed point $x^* = \left(1-\frac{1}{r}\right)$ is locally stable if $1 < r < 3$. It is unstable for $r > 3$. For $r = 1$, the fixed point $x^* = 0$ is non- hyperbolic fixed point. For $r = 1,3$, the fixed point $\left(1-\frac{1}{r}\right)$ is non-hyperbolic fixed point.

For $r < 1$ the parabola lies below the diagonal, and the origin is the only fixed point. As r increases, the parabola gets taller, becoming tangent to the diagonal at $r = 1$. For $r > 1$ the parabola intersects the diagonal in a second fixed point $x^* = 1 - \frac{1}{r}$ while the origin loses stability. Thus x^* bifurcates from the origin in a transcritical bifurcation at $r = 1$.

Note that at a bifurcation point fixed point of a map is non-hyperbolic.

Ex 5.8.1 Prove that the 2-cycle of logistic map is stable for $3 < r < 1 + \sqrt{6} = 3.449....$

Solution: To analyze the stability of a cycle, we have to find the stability of the fixed point. If p and q are fixed points of $1f^2(x) = x$, then p and q are fixed points of the second iterative map f^2. The original 2-cycle is stable precisely, if p and q are stable fixed point for f^2. A 2-cycle periodic fixed point will be stable if $|f'(p)f'(q)| < 1$. Here, $f(x) = rx(1 - x)$, $x \in [0,1]$, and $f'(x) = r - 2rx$. Now, for stability of 2-cycle gives

$$|f'(p)f'(q)| < 1$$

$$\Rightarrow |(r - 2rp)(r - 2rq)| < 1$$

$$\Rightarrow r^2|(1 - 2p)(1 - 2q)| < 1$$

$$\Rightarrow r^2|1 - 2(p + q) + 4pq| < 1$$ [p,q are fixed points of $f^2(x) = 0$, so that $p + q = \frac{r+1}{r}$ and $pr = \frac{r+1}{r^2}$]

$$\Rightarrow r^2 \left|1 - 2\frac{r+1}{r} + 4\frac{r+1}{r^2}\right| < 1$$

$$\Rightarrow |4 + 2r - r^2| < 1, \Rightarrow -1 < (r - 1)^2 - 5 < 1 \Rightarrow 4 < (r - 1)^2 < 6$$

$$\Rightarrow 3 < r < (1 + \sqrt{6}).$$

Hence proved.

5.9 Tent Map

The tent map is a one-dimensional piecewise linear map. Its graph resembles the front view of a tent.

The tent map is denoted by T(x) and defined as $T : [0,1] \to [0,1]$

$$T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1 - x), & \frac{1}{2} \le x \le 1. \end{cases}$$

The fixed point of the map obtained as,

if $0 \le x \le \frac{1}{2}$, $T(x) = x$, $\Rightarrow 2x = x \Rightarrow x = 0$.

If $\frac{1}{2} \le x \le 1$, $T(x) = x$, $\Rightarrow 2(1-x) = x \Rightarrow x = \frac{2}{3}$. Both fixed points are unstable as $f'(x) = 2(>1)$, $\forall x$.

The period-2 points of the map are obtained by solving the equation $T^2(x) = x$.

$$T^2(x) = ToT(x) = 2.2x = 4x \quad 0 \le x \le \frac{1}{2}$$

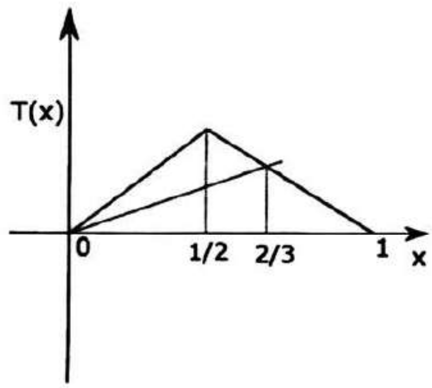
$$= 2(1 - 2(1 - x)) = 4x - 2 \quad \frac{1}{2} \le x \le 1.$$

$$\Rightarrow T^2(x) = 4x, \; 0 \le x \le$$

$$= -4x + 2, \; \frac{1}{4} \le x \le \frac{1}{2}$$

$$= 4x - 2, \; \frac{1}{2} \le x \le \frac{3}{4}$$

$$= 4 - 4x, \; \frac{3}{4} \le x \le 1.$$



The fixed points of T^2 are $T^2(x) = x \Rightarrow x = 0, \frac{2}{5}, \frac{2}{3}, \frac{4}{5}$

Again, $\left\{\frac{2}{5}, \frac{4}{5}\right\}$ is a two cycle because, $T\left(\frac{2}{5}\right) = 2.\frac{2}{5} = \frac{4}{5}$,

$T\left(\frac{4}{5}\right) = \frac{2}{5}$, and $T^2\left(\frac{2}{5}\right) = \frac{2}{5}$, $T^2\left(\frac{4}{5}\right) = \frac{4}{5}$.

Ex 5.9.1 Show that the Lipunov exponent $\lambda = \text{In } 2$ for the tent map, independent of the initial condition x_0.

Solution: Since $f'(x) = \pm 2$, $\forall x \in (0,1)$ except at $x = \frac{1}{2}$,

we find $\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \text{In } |f'(x_i)| = \log 2 > 0$. The Liapunov exponent λ is positive and so the tent map in Chaotic.

5.10 Horseshoe Map

The Smale horseshoe is the prototypical map possessing a chaotic invariant set. It is possible to create more complicated basin boundaries in dynamical systems.

Smale's horseshoe exhibited the chaotic behavior. The version of the horseshoe map H which we will analyze acts on the unit square by shrinking it three times vertically and stretching it three times horizontally, then bending it back 5 in the unit square as shown below in Figure 5.1. A horseshoe map generally is a diffeomorphism $F : R^2 \to R^2$ that maps a rectangle D over itself in the form of a horseshoe.

We study the dynamics of the map H on the set E of all points (x,j/) whose iterates, both forward and backward, stay in the unit square. Choosing the origin at the bottom left corner we get the following formula for H for any $(x, y) \in E$.

$$H(x, y) = \begin{cases} \left(3x, \frac{1}{3}y\right) \text{if } x \in \left[0, \frac{1}{3}\right] \\ (3-3x, 1-\frac{1}{3}y) \text{ if } x \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

The first component of H{x,y) is the tent map T(x). Assume for simplicity that homoclinic bifurcation causes a square ABCD (Figure 5.1) to be mapped in a particular manner, which will be specified, into a horseshoe ABCD.

The mapping is assumed to be carried out in this way: the square is stretched in the direction AD, compressed in the direction AB, bent through 180°, and placed back over the square (Figure 5.1). Suppose the mapping is repeated for such points as still lie in the square, and that this process is iterated. Figure 5.2 shows the first two iterations. The horizontal shaded strips in the square are chosen so as to map onto the vertical parts of the horseshoe. These two vertical strips now map onto the pair of thinner horseshoes. After two iterations there are points remaining in 16 'squares'; after the third iteration there will be 64 'squares', and so on. The limit set of the horseshoe map has a same limit set like Cantor set, but twodimensional structure. The implication is that there exists an uncountable number of points in the initial square which, when treated as initial states at t = 0 for iterated first returns, lead ultimately to endlessly

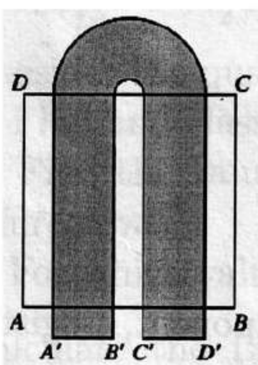


Figure 5.1: The horseshoe map acting on the unit square.

repeated scans of a certain set of points-the limit set-which constitutes the strange attractor. The associated oscillations will include periodic motions and bounded nonperiodic motions. The elements of this set are distributed on the unstable manifold since horseshoes can be constructed for each loop across the stable manifold.

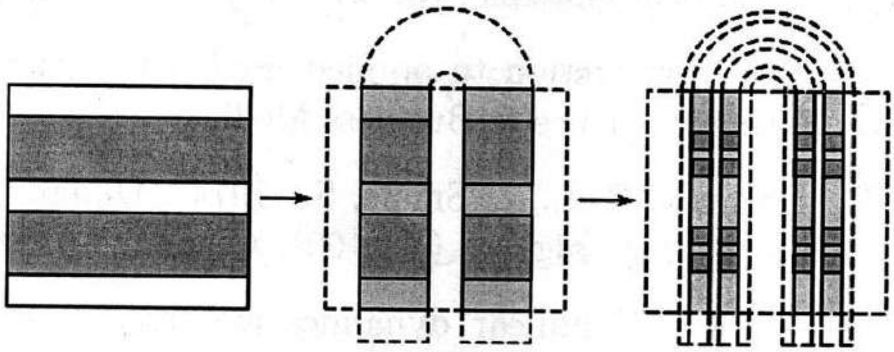


Figure 5.2: Successive horseshoe maps showing generation of the Cantor set.

5.11 Summary

In this unit, we have discussed the idea of discrete dynamical system. First we have discussed about fixed points, periodic points, periodic cycles and their stability. Here we only provided the idea about flip bifurcation which one of most significant bifurcation in discrete system.

Next we gave an brief sketch of sensitive dependence on initial condition of the system and Liapunov exponent. After defining Liapunov exponent we define chaos, most core theory of dynamical system. Then some examples of discrete dynamical system e.g logistic map, tent map and horseshoe map.

5.12 Keywords

Fixed point, Period-doubling bifurcation, Chaos, Liapunov exponent, Logistic map, Tent map, Horseshoe map.

5.13 Further Reading

1. Layek, G. C. (2015). An Introduction to Dynamical Systems and Chaos, Springer.
2. Verhulst, F. (2006). Nonlinear differential equations and dynamical systems. Springer Science & Business Media.

3. Jordan, D. W., & Smith, P. (1999). Nonlinear ordinary differential equations: an introduction to dynamical systems (Vol. 2). Oxford University Press, USA.

4. Wiggins, S. (2003). Introduction to applied nonlinear dynamical systems and chaos (Vol. 2). Springer Science &; Business Media.

5. Hirsch, M. W., Devaney, R. L., & Smale, S. (1974). Differential equations, dynamical systems, and linear algebra (Vol. 60). Academic press.

6. Strogatz, S. H. (2018). Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering. CRC Press.

7. Glendinning, P. (1994). Stability, instability and chaos: an introduction to the theory of nonlinear differential equations (Vol. 11). Cambridge university press.

8. Alligood, K. T., Sauer, T. D., & Yorke, J. A. (1996). Chaos (pp. 105-147). Springer New York.

5.14 Exercise

1. Define period-k points of a map with example.

2. Find the fixed points of following the maps and analyse its stability.

   a) $x_{n+1} = \sqrt{x_n}$

   b) $x_{n+1} = 2\dfrac{x_n}{1+x_n}$

   c) $x_{n+1} = 1 + \dfrac{1}{2}\sin x_n$

3. Consider the quadratic map $x_{n+1} = x_n^2 + c$.

   a) Find and classify all the fixed points as a function of c.

   b) Find the values of c at which the fixed points bifurcate, and classify those bifurcations.

   c) For which values of c is there a stable 2-cycle?

4. Calculate the Liapunov exponent for the decimal shift map $x_{n+1} = 10x_n (\text{mod } 1)$.

5. Prove that for logistic map, the first 2-cycle bifurcation occurs at $r = 3$.

6. a) Show that $f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$ is a map with a 3-cycle. Is it attracting or repelling?

   b) Consider the map $f(x) = -x^3$, $x \in R$. Show that the origin is an attracting fixed point and $\{-1,1\}$ is a repelling 2-cycle.

7. Consider the differential equation $\dot{x} = x - x^3 - b\sin(2\pi t)$ where $|b|$ is small. What can you say about solutions of this equation? Are there any periodic solutions?

8. Find the Liapunov exponent for the logistic map. Also, calculate the Liapunov exponent for $x_{n+1} = r\sin(\pi x_n)$, $r > 0$, $x_n \in [0,1]$.

9. Prove that the map g : unit circle $s \rightarrow s$ defined by $g(\theta) = \theta + \alpha$, where the rotation α is irrational is topologically transitive.

10. Show that $T^n(x)$ of the tent map $T(n)$ has 2^n fixed points.

11. Show that there must exist three distinct periodic-4 orbits of $T^4(x)$ of the tent map $T(x)$.

12. Suppose a continuous map $f : I \rightarrow I$ has a horseshoe. Then prove that (i) f^k has at least 2^k fixed points, (ii) f has periodic points of every period (iii) f has an uncountable number of aperiodic orbits.

Notes